This is a digital copy of a book that was preserved for generations on library shelves before it was carefully scanned by Google as part of a project to make the world's books discoverable online.

It has survived long enough for the copyright to expire and the book to enter the public domain. A public domain book is one that was never subject to copyright or whose legal copyright term has expired. Whether a book is in the public domain may vary country to country. Public domain books are our gateways to the past, representing a wealth of history, culture and knowledge that's often difficult to discover.

Marks, notations and other marginalia present in the original volume will appear in this file - a reminder of this book's long journey from the publisher to a library and finally to you.

Usage guidelines

Google is proud to partner with libraries to digitize public domain materials and make them widely accessible. Public domain books belong to the public and we are merely their custodians. Nevertheless, this work is expensive, so in order to keep providing this resource, we have taken steps to prevent abuse by commercial parties, including placing technical restrictions on automated querying.

We also ask that you:

+ *Make non-commercial use of the files* We designed Google Book Search for use by individuals, and we request that you use these files for personal, non-commercial purposes.

+ *Refrain from automated querying* Do not send automated queries of any sort to Google's system: If you are conducting research on machine translation, optical character recognition or other areas where access to a large amount of text is helpful, please contact us. We encourage the use of public domain materials for these purposes and may be able to help.

+ *Maintain attribution* The Google "watermark" you see on each file is essential for informing people about this project and helping them find additional materials through Google Book Search. Please do not remove it.

+ *Keep it legal* Whatever your use, remember that you are responsible for ensuring that what you are doing is legal. Do not assume that just because we believe a book is in the public domain for users in the United States, that the work is also in the public domain for users in other countries. Whether a book is still in copyright varies from country to country, and we can't offer guidance on whether any specific use of any specific book is allowed. Please do not assume that a book's appearance in Google Book Search means it can be used in any manner anywhere in the world. Copyright infringement liability can be quite severe.

About Google Book Search

Google's mission is to organize the world's information and to make it universally accessible and useful. Google Book Search helps readers discover the world's books while helping authors and publishers reach new audiences. You can search through the full text of this book on the web at `http://books.google.com/`

Tr 01: 288.86

SCIENCE CENTER LIBRARY

FROM THE BEQUEST OF

HORACE APPLETON HAVEN,

OF PORTSMOUTH, N. H.

(Class of 1842.)

Digitized by Google

Digitized by Google

Digitized by Google

MATHEMATICAL

QUESTIONS AND SOLUTIONS

FROM "THE EDUCATIONAL TIMES."

WITH MANY

PAPERS AND SOLUTIONS

IN ADDITION TO THOSE

PUBLISHED IN "THE EDUCATIONAL TIMES."

EDITED BY

CONSTANCE I. MARKS, B.A.

NEW SERIES.

VOLUME VII.

LONDON:

FRANCIS HODGSON, 89 FARRINGDON STREET, E.C.

1905.

Digitized by Google

Math 388.86

HARVARD COLLEGE
JUL 1 1905
LIBRARY.

Haven fund.
(VII.)

HARVARD COLLEGE
LIBRARY

Digitized by Google

CONTRIBUTORS.

ABBOTT, R. C., B.A.; Marlborough College.
ADAIR, E. W.; Guernsey.
ALEXANDER, J. J., M.A.; Tavistock.
ALLEN, Rev. A. J. C., M.A.; Cambridge.
ANDERSON, Prof., M.A.; Queen's Coll., Galway.
ANTHONY, EDWYN, M.A.; Westwood, Hereford.
ANTONIO, J.; Penang.
APPLETON, Miss E.; Liverpool.
ARCHIBALD, R. C., M.A., Ph.D.; Sackville, N.B., Canada.
ARMENANTE, Professor; Pesaro.
ARNOLD, I.; Institution, Belfast.
AYRTON, Mrs.; Hyde Park, W.
BAGCHI, S. C., B.A.; Cambridge.
BALL, Sir ROBERT STAWELL, LL.D., F.R.S., Professor of Astronomy and Geometry, Camb.
BALL, W. W. ROUSE; Cambridge.
BARNIVILLE, JOHN J., B.A.; Chittagong, Bengal.
BARRETT, E. P., B.A.; Stroud.
BARTON, W. J., M.A.; Highgate, London.
BATEMAN, H. B.A.; Cambridge.
BATES, G.N., M.A.; Ramsgate.
BATTAGLINI, Prof. GIUSEPPE; Univ. di Roma.
BEARD, W. F., M.A., Melbourne.
BELL, A. H.; Litchfield, Ill., U.S.A.
BEMAN, Prof. W. W.; Univ. Michigan.
BESANT, W. H., D.Sc., F.R.S.; Cambridge.
BEYENS, Prof. IGNACIO, Lt.-Col. du Génie; Cadiz.
BHATTACHARYYA, Prof. N.
BHUT, Professor ATH BIJAH, M.A.; Dacca.
BICKERDIKE, C.; Scarborough.
BIDDLE, D. (Editor, Vols. LXVII.–LXXV.); Charlton Lodge, Kingston-on-Thames.
BIGGS, R., M.A., LL.D.; Portora, Enniskillen.
BILLUPS, H. B.; Sandown, I. of W.
BIRCH, Rev. J. G., M.A.; Bunralty, co. Clare.
BIRTWISTLE, G., B.A., B.Sc.; Pemb. Coll., Camb.
BLAIKIE, J., M.A.; Norwood.
BLATER, JOSEPH; Baden Baden.
BLYTHE, W. H., M.A.; St. Mark's Lodge, Newnham.
BOLAND, J.; Sand Hill, Gibraltar.
BOOTH, Prof. W.; Hoogly College, Bengal.
BORDAGE, Prof. EDMOND; Coll. de Nantua.
BOURNE, Rev. C. W., M.A.; King's Coll. Sch.
BOUTON, V. J., B.Sc.; Hampton Wick.
BOWSER, Prof. E. A.; New Jersey, U.S.A.
BOYS, Prof.; Royal College of Science.
BRILL, J., M.A.; Cambridge.
BROCARD, H., Chef de Bataillon; Bar-le-Duc.
BROOKS, Prof. E.; Millersville, Pennsylvania.
BROWN, Prof.; Providence, U.S.A.
BROWN, A. CRUM, D.Sc.; Edinburgh.
BROWN, Prof. JOSEPH; Dublin.
BRUCE, Rev. ROBERT, D.D.; Huddersfield.
BRUNYATE, W. E.; Trinity Coll., Cambridge.
BRYAN, G. H., F.R.S.; Bangor.
BRYANT, R., D.Sc., B.A.; London Institution.
BUCK, EDWARD, M.A.; Oakhill, Bath.
BURKE, J., M.A.; Trinity College, Dublin.
BURNSIDE, Professor, M.A.; Univ. of Dublin.
BURSTALL, H. J. W.; King's Coll., London.
CAMA, B. N., M.A.; St. John's Coll., Cambridge.
CAMPBELL, J. E., M.A.; Hertford Coll., Oxford.
CANHAM, D.; Norwich.
CAPEL, H. N., LL.B.; Bedford Square, London.
CARMODY, W. P., B.A.; Clonmel Gram. School.
CARR, G. S., M.A.; Chavey Down, Bracknell, Berks.
CATALÁ, Prof. L.
CAVALLIN, Prof., M.A.; University of Upsala.
CAVE, A. W., B.A.; Magdalen College, Oxford.
CAVE, F. E.; Girton Coll., Cambridge.
CHAKRAVARTI, Professor BYOMAKESA, M.A.; Calcutta.
CHANDRA BASÚ, Professor, M.A.; Calcutta.
CHAPMAN, Prof. R. W., M.A.; Univ. of Adelaide.
CHARTRES, R.; 72 Carter Street, Manchester.
CHRISTIE, R. W. D.; Stanley Coll., Liverpool.
CLARK, A.; Aberystwyth.
CLARKE, Colonel A. R., C.B., F.R.S.; Redhill.
CLAYTON, Prof., M.A.; Parkville, Melbourne.
CLIFFORD, Prof. W. K., F.R.S.
COCHEZ, Professor; Paris.
COHEN, ARTHUR, M.A., K.C., M.P.; Holland Pk.
COLSON, C. G., M.A.; University of St. Andrews.
CONSTABLE, Rev. W. J., M.A.; Uppingham.
COONDOO, Prof. MAHENDRA NATH; Benares.
COONEY, W. S.; High School, Dublin.
COOPER, T. M. A., M.A.; Cheltenham.
COTTERILL, J. H., M.A.; St. Alban's Mansions, W.
CRAWFORD, G. E., M.A.; Clifton, Bristol.
CREMONA, Prof. LUIGI; Rome.
CROFTON, Professor M. W., F.R.S.; Ryde.
CROKE, J. O'BYRNE, M.A.; Dublin.
CULLEN, Rev. J.; Manresa House, Roehampton, S.W.
CULLEY, Prof., M.A.; St. David's Coll., Lampeter.
CUNNINGHAM, ALLAN, Lt.-Col., R.E.; Kensington
CURJEL, H. W., M.A.; Southport.
CURTIS, Prof., M.A., S.J.; Univ. Coll., Dublin.
CZUBER, Dr. EMANUEL; Prag, Bohemia.
DALLAS, R. J.; Pemberton Gardens, N.
DANIEL, VINCENT, B.Sc.; Cardiff.
DANIELS, A. E., B.A.; Nottingham.
DARBOUX, Professor; Paris.
DATA, Prof. PROMATHANATH, M.A.; Delhi.
DAVIES, Prof., Brown Univ., Providence, U.S.A.
DAVIS, Rev. B., M.A.; Principal of Jai Narayan College, Benares.
DAVIS, R. F., M.A.; Acton Green.
DAWSON, H. G., M.A.; Sheffield.
DE LONGCHAMPS, Professor; Paris,
DE VRIES, Professor JAN; Utrecht.
DE WACHTER, Professor, M.A.; Schaarbeck.
DEY, Prof. NARENDRA LAL, M.A.; Calcutta.
DIBB, J. H., B.Sc.; Hendon.
DICK, Hon. G. R., M.A.; Auditor Gen., Mauritius.
DICKSON, J. D. H., M.A.; Peterhouse, Camb.
DICKSON, W.; Edinburgh.
DOBBS, W. J., M.A.; Earl's Court.
D'OCAGNE, MAURICE; Paris.
DROZ-FARNY, Prof. A.; Collège de Porrentruy.
DRURY, H. D., M.A.; Marlborough College.
DUPAIN, J. C.; Professor au Lycée d'Angoulême.
DURELL, Rev. C. V., M.A.; Fulbourn Rectory, near Cambridge.
EASTON, BELLE, B.Sc.; Lockport, New York.
EDMONDSON, Prof. T. W.; New York Univ.
EDWARDES, DAVID, B.A.; Bangor.
ELLIOTT, Prof. E. B., F.R.S.; F. Mag. Coll., Oxon.; Oxford.
ELTON, E. F., M.A.; Wellington College.
EMMERICH, Prof., Ph.D.; Mülheim-am-Ruhr.
ESCOTT, Prof. E. B.; Univ. of Michigan, U.S.A.
EVE, A. E., M.A.; Marlborough College.
EVERETT, Prof. J. D., D.C.L.
EWIN, CECIL; Drumcondra, Dublin.
FINKEL, B. F.; Missouri, U.S.A.
FLOOD, P. W.; Drumcondra, Dublin.
FORSYTH, Prof., Sc.D.; Cambridge.
FORTEY, H., M.A.
FOSTER, F. W., B.A.; Chelsea,

Digitized by Google

FOSTER, Prof. G. CAREY, F.R.S.
FOSTER, W. S.; Hoddesdon, Herts.
FRANKLIN, Prof. CHRISTINE LADD, M.A., B.Sc.
FREETH, Rev. Dr.
GALLIERS, Rev. T., M.A.; Kirstead Rect., Norwich
GASKIN, C. H.; Birkenhead.
GENESE, Prof., M.A.; Univ. Coll., Aberystwyth.
GEORGI, MAURICE; Paris.
GERRANS, H. T., M.A.; Fel. & Tut. Wor. Coll., Oxon.
GHOSH, SRISH CHANDRA, M.A.; Calcutta.
GHOSH, Prof. U. CHANDRA; Allahabad.
GIBSON, T. PAUL; St. Cloud, France.
GLAISHER, J. W. L., F.R.S.; Fel. Trin. Coll., Camb.
GOLDENBERG, Professor, M.A.; Moscow.
GOPALACHARI, Prof.; Presidency Coll., Madras.
GORDON, A., M.A.; Barnwood Ho., Gloucester.
GOULD, S. C.; New Hampshire, U.S.A.
GRACE, G. M., B.A., B.Sc.; Hammersmith.
GRACE, J. H.; Peterhouse, Cambridge.
GRAHAM, R. A., M.A.; Trinity College, Dublin.
GRAHAM, THOMAS; Kilkeel, Co. Down.
GREENHILL, A.G., M.A., F.R.S.; Artil.C., W'wich.
GREENSTREET, W. J., M.A.; Marling Sch., Stroud.
GREENWOOD, JAMES M.; Kansas City.
GRIFFITH, GERTRUDE, B.A.; Southampton.
GRIFFITHS, D. T., M.A.; Caermarthen.
GRIFFITHS, G. J., M.A.; Fellow Ch. Coll., Camb.
GRIFFITHS, J., M.A.; Fellow of Jesus Coll., Oxon.
GROSS, W., LL.D.; Bournemouth.
GROVE, W. B., B.A.; Perry Bar, Birmingham.
GUNNING, S., J.P.; Cookstown, Co. Tyrone.
HADAMARD, Professor, M.A.; Paris.
HAIGH, E., B.A., B.Sc.; King's Sch., Warwick.
HALL, Professor ASAPH, M.A.; Washington.
HAMILTON, J. P.; Penang.
HAMMOND, J., M.A.; 14 Granville Square, W.C.
HANUMANTA RAU, Professor, B.A.; Madras.
HARDY, G. H., M.A.; Trinity Coll., Cambridge.
HARKEMA, C.; University of St. Petersburg.
HARLEY, Rev. R., M A., F.R.S.; Forest Hill.
HATTON, J. L. S.; E. Lond. Tech. Coll.
HAY, THEODORE W.; Leven, Fife.
HEAL, W. E., M.A.; Marion, Ind., U.S.A.
HENDERSON, R., B.A., Queen's Coll., Belfast.
HEPPEL, GEORGE, M.A.; Ealing.
HERVEY, F. R. J., M.A.; Worthing.
HILL, Rev. E., M.A.; Cockfield, Suffolk.
HILL, F. W.; City of London School.
HILL, Prof. M. J. M., M.A., D.Sc., F.R.S.; Univ. Coll. Lond.
HILLHOUSE, Dr. W.; Newhaven, Conn., U.S.A.
HILLS, G. F. S.; Trin. Coll., Camb.
HILLYER, C. E., M.A.; Cambridge.
HIME, Lt.-Col.; Alexandria.
HINTON, C. H., M.A.; Cheltenham College.
HIPPISLEY, R. L., Major, R.E.; Farnborough.
HOBSON, Dr.; Cambridge.
HOLT, J. R., M.A.; Dublin.
HOOKER, J. H., M.A.; 184 Evering Road, N.E.
HOPKINS, Rev. G. H., M.A.; Stratton, Cornwall.
HOROBIN, C. J., B.A.; M.L.S.C.
HOWSE, G. F.; Balliol College, Oxford.
HUDSON, C. T., LL.D.; Manilla Hall, Clifton.
HUDSON, J. F.; Gunnersbury House, Acton.
HUDSON, Prof. W. H. H., M.A.; 15 Altenburg Gardens, S.W.
HUGHES, EDITH K. A.; Stroud.
HUGHES, W. R.; Jesus College, Cambridge.
HUNTINGTON, E.V., Ph.D.; Camb., Mass., U.S.A.
IKIN, A. E., B.Sc.; Kettering.
ISSERLIS, L., B.A.; Munic. Tech. Inst., West Ham.
JACKSON, Rev. F. H., R.N., M.A.; H.M.S. "Irresistible," Malta.
JACKSON, Miss F. H.; Towson, Baltimore.
JACKSON, W. H., B.A.; Clare Coll., Camb.
JACOBS, H. J.; London Institution, Finsbury.
JAGO, G.; Plymouth Public School.
JENKINS, MORGAN, M.A.; Tunbridge Wells.
JOHNSON, Prof., M.A.; Annapolis, Maryland.
JOHNSTONE, W.J., M.A.; Univ. Coll., Aberystwyth
JOLLIFFE, A. E., M.A.; St. David's College.
KAHN, A., B A.; Mildmay Grove, N.
KALIPADU BASÔ, Prof., M.A.; Dacca College.
KATHEEN, R. P., M.A.; Fleetwood, Bath.
KELVIN, Lord, F.R.S.; Largs, Ayrshire.
KENNEDY, D., M.A.; Catholic Univ., Dublin.
KITCHIN, F. L., B.A., R.N.; Exmouth.
KITCHIN, Rev. J. L., M.A.; Exmouth.
KITTUDGE, LIZZIE A.; Boston, United States.
KLEIN, Professor FELIX; Univ. Göttingen.
KNISELEY, ALEX.; Columbia City, Indiana.
KNOWLES, R., L.C.P.; Tottenham.
KOEHLER, J.; Rue St. Jacques, Paris.
KOLBE, Rev. Dr.; St. Mary's Coll., Cape Town.
KRISHMACHANDRA DE, Prof.; Pudukkottai.
KRISHNAMACHARRY, Prof., M.A.; Tiupati, India.
LACHLAN, R., M.A.; Newnham Terrace, Camb.
LAISANT, C. A.; Paris.
LAMB, E.; George Town, Penang.
LAMPE, Prof., Ed. of *Jahrb. der Math.*; Berlin.
LANGLEY, E. M., B.A.; Adelaide Sq., Bedford.
LASKER, E., Ph.D.; Manchester.
LAUVERNAY, Prof.; Paris.
LAVERTY, W. H., M.A.; late Exam. in Univ. Oxford.
LAWRENCE, E. J.; Ex-Fell. Trin. Coll., Camb.
LAWSON, J. S., M.A.; Newcastle-on-Tyne.
LAYNG, A. E.; Grammar School, Stafford.
LEGABEKE, Professor; Delft.
LEIDHOLD, R., M.A.; Finsbury Park.
LEMOINE, E.; Avenue du Maine, Paris.
LEUDESDORF, C., M.A.; Fel. Pembroke C., Oxon.
LEWIS, JOHN; Cardiff.
LEWIS, WM., M.A.; Headmaster, Llandyssul.
LIAGRE, General; Rue Carotz, Brussels.
LONDON, Rev. H., M.A.; Wimbledon.
LONGHURST, Rev. E. S., B.A.; Newark.
LOWRY, W. H., M.A.; Blackrock, Dublin.
LOXTON, C. A.; Cannock.
LUDWIG, Professor; North Carolina, U.S.A.
MCALISTER, DONALD, M.A., D.Sc.; Cambridge.
MACAULAY, F. S., M.A.; St. Paul's School.
MCCLELLAND, W. J., B.A.; Prin. of Santry School.
MACCOLL, HUGH, B.A.; Boulogne.
MCCOY, Rev. R.; Stonyhurst.
MCCUBBIN, J., B.A.; Burgh Academy, Kilsyth.
MACDONALD, W. J., M.A.; Edinburgh.
MACFARLANE, Prof. A., D.Sc.; Univ. of Texas.
MCINTOSH, ALEX., B.A.; Bedford Row, London.
MACKAY, J. S.; Edinburgh.
MACLEOD, J., H.M.I.S., I.S.O.; Elgin.
MACMAHON, Major P. A., F.R.S.; R.M.Academy.
MCMAHON, Prof., M.A.; Cornell Univ., Ithaca.
MACMURCHY, A., B.A.; Univ. Coll., Toronto.
MCVICKER, C. E., M.A.; Kensington.
MADDISON, ISABEL, B.A.; Brynmawr, Pa., U.S.A.
MADHAVARAO, Prof., M.A.; Vizianagram.
MAINPRISE, B. W.; R. Naval School, Eltham.
MALET, J. C., M.A., F.R.S.; Dublin.
MANN, M. J. J.; Cromwell Road, London.
MANNHEIM, M.; Prof. à l'École Polytech. Paris.
MARKS, Miss CONSTANCE I., B.A. (Editor); W. Kensington.
MARTIN, ARTEMAS, M.A., Ph.D.; Washington.
MASSEY, W. H.; Twyford, Berks.
MATHEWS, Professor G. B., M.A.; St. John's Coll., Cambridge.
MATZ, Prof., M.A.; King's Mountain, Carolina.
MELLOR, J. W., B.Sc.; Owens Coll., M'chester.
MERRIFIELD J., LL.D., F.R.A.S.; Plymouth.
MERRIMAN, MANSFIELD, M.A.; Yale College.
METCALFE, G., M.A.; Wimbledon.
MEYER, MARY S.; Girton College, Cambridge.
MILNE, Rev. J. J., M.A.
MINCHIN, G. M., M.A.; Prof. in Coopers Hill Coll.
MITCHELL, HUGH, M.A.; Edinburgh.
MITCHESON, Rev. T., B.A.
MITTAG-LEFFLER, Professor; Stockholm.
MONCK, H. ST., M.A.; Trin. Coll., Dublin.
MOORE, H. K., B.A.; Trin. Coll., Dublin.
MOREL, Professor; Paris.
MORGAN, C., M.A.; R. Naval Coll., Greenwich.
MORLEY, Prof., Sc.D.; Johns Hopkins Univ., Baltimore.
MORRICE, G. G., M.A., M.D.; Weymouth.
MUGGERIDGE, G. D., B.A.; Pemb. Coll., Camb,

Digitized by Google

MUIR, THOMAS, M.A., F.R.S.; Cape Town.
MUIRHEAD, R. F.; Glasgow.
MUKHOPADHYAY, Prof. ASUTOSH, M.A., F.R.S.E.
MUKHOPADHYAY, Prof. SYAMADAS; Chinsurah.
MULCASTER, J. W., M.A.; Allendale Town.
MUNINDRA, NATH RAY, Prof. M.A., LL.B.
NANSON, Prof. E. J.; Melbourne University.
NEALE, C. M.; Middle Temple, E.C.
NESBITT, A. M., M.A.; Melbourne.
NESBITT, H. A.; Hampstead.
NEUBERG, Professor; University of Liège.
NEWCOMB, Prof. SIMON, M.A.; Washington.
NISNER, I.; Westbourne Grove, W.
NIXON, R. C. J., M.A.
NORTH, FRANCIS C.; Ashton-on-Mersey.
O'CONNELL, Major-General P.; Cheltenham.
OLDHAM, C. H., B.A., B.L.; Dublin.
OPENSHAW, Rev. T. W., M.A.; Clifton, Bristol.
ORCHARD, JOHN, M.A.
ORCHARD, Professor, M.A., B.Sc.; Hampstead.
O'REGAN, JOHN; New Street, Limerick.
ORPEUR, HERBERT; East Dulwich.
OWEN, J. A., B.Sc.; Tennyson St., Liverpool.
PANCHAPAGESA AIYAR, K.G.; Madras.
PANTON, A. W., M.A.; Fell. of Trin. Coll., Dublin.
PARANJPYE, R. P., B.A.; Poona, India.
PAROLING, Professor; Cornell University.
PEACHELL, F. H., B.A.; High Wycombe.
PEELE, T., A.I.A.; Manchester.
PEIRCE, Prof. B. O.; Harvard College.
PEIRIS, M. R.; Moratuwa, Ceylon.
PERRIN, EMILY, B.Sc.; St. John's Wood Park.
PHILLIPS, F. B. W.; 7 Harpur Place, Bedford.
PILLAI, Professeur, M.A.; Trichy., Madras.
PLAMENEWSKI, H., M.A.; Dahgestan.
POCKLINGTON, H. C., M.A.; Yorks Coll., Leeds.
POLIGNAC, Prince CAMILLE DE.
POLLEXFEN, H., B.A.; Windermere College.
POOLE, A. W.; London.
POOLE, GERTRUDE, B.A.; Cheltenham.
PORTELA, Señor FRANCISCO DE V.; Cadiz.
PORTER, FRANK, M.A.
PRESCOTT, J., B.A.; Manningham Lane, Bradf'd.
PRESSLAND, A. J., M.A.; Academy, Edinburgh.
PRESTON, G. W., B.A.; Wheatley, near Halifax.
PRICE, H., M.A.; Penarth.
PRUDDEN, FRANCES E.; Lockport, New York.
PUCKLE, G. HALE, M.A.; Windermere.
PURSER, FREDK., M.A.; Fell. Trin. Coll., Dublin.
PURSER, Prof., M.A.; Queen's Coll., Belfast.
PUTNAM, K. S., M.A.; Rome, New York.
PYDDOKE, H. W., M.A.; Oxford.
QUIN, A. G. B.; Streatham.
QUINT, Dr.; Gravenhäge, Netherlands.
RADHAKRISHNAN, Prof., M.A.; Pudukkottai.
RAMA AIYANGAR, Prof.; Trichinopoly, Madras.
RAMACHANDRA ROW, Prof., M.A.; Trichinopoly.
RAMSEY, A. S., M.A.; Fettes Coll., Edinburgh.
RANGACHARI, V. SRINIVASA, B.A.; Government College, Kumbakonum, India.
RAU, ALBERT G., B.S.; Bethlehem, Pa., U.S.A.
RAU, K. DEVA, B.A.; Mangalore.
RAU, V. RAGUNATH; Torupati, India.
RAWSON, ROBERT; Havant, Hants.
REAY, LIONEL E., B.A.; Canterbury.
REES, E. W.; Penarth, Cardiff.
REEVES, F. W., B.A.
REEVES, G. M., M.A.; Lee, Kent.
REYNOLDS, B., M.A.; Cambridge.
RICE, JAMES; Craigie Road, Belfast.
RICHARDS, DAVID, B.A.; Christ's Coll., Brecon.
RICHARDSON, A. R.
RICHARDSON, Rev. G., M.A.; Winchester Coll.
ROACH, Rev. T., M.A.; Twyford, Hants.
ROBB, A. A.; St. John's Coll., Cambridge.
ROBERTS, A. D., B.A.; Jesus Coll., Oxford.
ROBERTS, R. A., M.A.; New York.
ROBERTS, W. R. W., M.A.; Fell. of Trin. Coll., Dub.
ROBERTSON, T. BRAILSFORD; S. Australia.
ROBSON, H. C., B.A.; Sidney Sussex Coll., Camb.
ROGEL, Professor; Mittweida, Germany.
ROGERS, L. J., M.A.; Yorkshire Coll., Leeds.
ROSS, C. M.; Putney, S.W.
ROWTON, Rev. S. J., M.A., Mus.Doc.; Folkestone.
ROY, Professor KALIPRASANNA, M.A.; Agra.
RUGGERO, SIMONELLI; Università di Roma.
RUSSELL, ALEXANDER, B.A.; Oxford.
RUSSELL, B.; Pembroke Lodge, Richmond.
RUSSELL, J. W., M.A.; Merton College, Oxford.
RUSSELL, R., M.A.; Fell. of Trin. Coll., Dublin.
RUSSELL, T.; Caius College, Cambridge.
RUTTER, E.; Sunderland.
SADLER, JOHN; Birkenhead.
ST. CLAIR, J. C., M.A.; Ryder Street, London.
SALMON, Rev. GEORGE, D.D., F.R.S.
SALMON, W. H., B.A., B.Sc.; Bedford.
SANDBERG, C. F., M.A.; Oxford.
SANDERSON, Rev. C. M., M.A.; Brington Rectory.
SANJANA, Prof., M.A.; Bhavnagar, Bombay.
SARADARANJAN RAY, Prof., M.A.; Dacca.
SARKAR, Prof. BENI MADHAV, M.A.; Agra.
SARKAR, Prof. NILKANTHA, M.A.; Calcutta.
SAVAGE, Prof. T.
SCHARAR, C. H.; Scranton, U.S.A.
SCHEFFER, Professor; Mercersbury Coll., Pa.
SCHOUTE, Prof. P. H.; University of Groningen.
SCOTT, A. W., M.A.; St. David's Coll., Lampeter.
SCOTT, Professor CHARLOTTE A., D.Sc.
SCOTT, GEORGE, M.A.; Greystones.
SCOTT, R. F., M.A.; Fell. St. John's Coll., Camb.
SCRIMGEOUR, W., M.A., B.Sc.; Peebles.
SEARLE, C.
SEGAR, HUGH W.; Trinity Coll., Cambridge.
SEN, Prof. RAJ. MOHAN; RajshabyeColl., Bengal.
SEWELL, Rev. H., B.A.; Bury Grammar School.
SEYMOUR, W. R., M.A.; St. Leonards.
SHARPE, J. W., M.A.; Bournemouth.
SHEPHERD, Rev. A. J. P., B.A.: Fellow of Queen's College, Oxford.
SHIELDS, Prof., M.A.; Coopwood, Mississippi.
SIMMONS, Rev. T. C., M.A.; Grainthorpe Vicarage, North Thoresby S. O., Lincs.
SIRCOM, Prof. SEBASTIAN, M.A.; Liverpool.
SITARAMAIYAR, Prof., Hindu Coll., Tinnevelly.
SIVERLEY, WALTER; Oil City, Pennsylvania.
SKRIMSHIRE, Rev. E., M.A.; Llandaff.
SMITH, C., M.A.; Sidney Sussex College, Camb.
SMITH, Professor D. E.: Brockport, N.Y.
SMITH, J. G.; Portora, Enniskillen.
SMYLY, J. G.: Merrion Square, Dublin.
SOPER, H. E., B.A.; Highgate.
SPARK, Rev. F. M.; Lewisham, London.
STANHAM, W. C., B.A.; Kensington.
STEEDE, B.H., M.A.; Trinity College, Dublin.
STEGGALL, Prof. J. E. A., M.A.; Dundee.
STEPHEN, ST. JOHN, B.A.: Caius Coll., Cambridge.
STEWART, H., M.A.; Islington.
STOOPS, J. M., B.A.; Victoria College, Belfast.
STOOPS, W.; Newry.
STORMONTH, JAMES; Edinburgh.
STORR, G. G., M.A.: Clerk of the Medical Council.
STOTT, WALTER, B.A.; Royal Ins. Co., Liverpl.
STRACHAN, J. C.; Anerley, London, S.E.
STUART, T., D.Sc.; Belfast.
SWAMINATHA AIYAR, Professor, M.A.; Madras.
SYMONS, E. W., M.A.; Grove House, Walsall.
TANNER, Prof. H. W. L., M.A.; S. Wales Univ. Coll.
TARLETON, F. A., M.A.; Fell. Trin. Coll., Dublin.
TATE, JAMES, M.A.; Kedleston Road, Derby.
TAYLOR, Rev. CHARLES, D.D.; Master of St. John's College, Cambridge.
TAYLOR, F. G., M.A., B.Sc.; Nottingham.
TAYLOR, H. M., M.A., F.R.S.; Fellow Trin. Coll., Cambridge.
TAYLOR, J. H., M.A.; Little Trinity, Cambridge.
TAYLOR, W. W., M.A.; Oxford.
TERRY, Rev. T. R., M.A.; Ilsley Rectory.
THIRD, J. A., M.A.; Beith, Ayrshire.
THOMAS, A. E., M.A.; Merton College, Oxford.
THOMAS, Rev. D., M.A.; Garsington Rect., Oxford.
THOMPSON, A. W.
THOMPSON, C. H.; St. David's Coll., Lampeter.
THOMSON, Rev. F. D., M.A.; Ex-Fellow of St. John's Coll., Cambridge.
THOMSON, W. L.; Edinburgh.
THYAGARAGAIYAR, V. R., M.A.; Bangalore.

Digitized by Google

TIPPLE, E. F.; Thomason Coll., Roorkee.
TIRELLI, Dr. FRANCESCO; Univ. di Roma.
TODD, E. WALTER; Belfast.
TOMBE, A. S., M.A.; Tech. Schools, Plymouth.
TRACHTENBERG, H. L.; Trin. Coll., Camb.
TRACHTENBERG, M. I., St. John's Coll., Camb.
TRAILL, ANTHONY, M.A., M.D.; Fell. T.C.D.
TUCKER, R., M.A.; Worthing.
VAN DER HEYDEN, A. F., M.A.; Middlesbrough.
VENKATA, Prof. K. RAMANAIYA; Madras.
VIGARIÉ, ÉMILE; Laissac, Aveyron.
VINCENZO, JACOBINI; Università di Roma.
VOSE, Professor G. B.; Washington.
WALENN, W. H.; Mem. Phys. Society, London.
WALKER, G. F.; Queens' College, Cambridge.
WALLIS, B. C., F.C.P.; Aldridge, Staffs.
WALMSLEY, J., B.A.; Eccles, Manchester.
WARBURTON-WHITE, R., B.A.; Salisbury.
WARD, BEATRICE A., B.Sc.; Cheltenham.
WARD, F. L., B.A.; Hartlepool.
WARREN, A. T., M.A.; Usk Terrace, Brecon.
WARREN, R., M.A.; Trinity College, Dublin.
WASSON, J. C.; Kells, Meath.
WATHERSTON, Rev. A. L., M.A.; Bowdon.
WATSON, D., M.A.; Folkestone.
WATSON, Rev. H. W.; Ex-Fell. Trinity Coll., Camb.
WATTS, J. O.; Middlesbrough.
WEBB, G. H.; Halifax.
WEBB, H. A.; Trinity College, Cambridge.
WEBSTER, H. W., M.D.; Chelsea.
WELLACOT, Rev. W. T., M.A.; Newton Abbot.
WERTSCH, FRANZ; Weimar.
WEST, J. WOODCOTE; New Cross.
WHALLEY, L. J., B.Sc.; Leytonstone.
WHAPHAM, R. H. W., M.A.; Manchester.
WHITE, E. A., M.A.; Eckington, Pershore.
WHITEHEAD, R. F., B.A.; London.
WHITESIDE, G. M.A.; Eccleston, Lancashire.
WHITWORTH, Rev. Preb. W. A., M.A.; London.
WIGGINS, Rev. T., B.A.; St. Beuno's Coll., St. Asaph,
WILCOX, Rev. A. M., M.A.; Branhope, Leeds.
WILKINSON, J. F.; Bacup.
WILLIAMS, A. E., M.A.; Texas.
WILLIAMS, D. J., M.A.; Worcester Coll., Oxon.
WILLIAMSON, B., M.A.; F. & T. Trin. Coll., Dub
WILMOT, Rev. W. T., M.A.; Newton Abbot.
WILSON, Rev. J., M.A.; Bannockburn.
WILSON, Ven. Archdeacon J. M., M.A., F.G.S.
WILSON, Rev. J. R., M.A.; Royston, Cambs.
WOLSTENHOLME, Professor, M.A., Sc.D.
WOLTERS, J. B.; Groningen.
WOODALL, H. J., A.R.C.S.; Stockport.
WOODCOCK, T., B.A.; Twickenham.
WORKMAN, W. P., M.A.; Bath.
WORTHINGTON, W. R., M.A.; Southport.
WRIGHT, W. J.; New Jersey, U.S.A.
YOUNG, JOHN, B.A.; Portadown, Ireland.
YOUNGMAN, C. E., M.A.; Corinth House, Cheltenham.
ZERR, Professor, M.A.; Staunton, Virginia.

*** Of this Mathematical publication there have now been issued the seventy-five Volumes of the Original Series and seven Volumes of the New Series. Each Volume contains, in addition to the papers and solutions that have appeared in *The Educational Times*, an equal quantity of new matter, consisting of further papers and solutions, and of completions of, or additions to, papers and solutions that have been partly published in the pages of the journal.

These volumes contain contributions, in all branches of Mathematics, from most of the leading Mathematicians in this and other countries.

New Subscribers may have any of these Volumes at Subscription prices.

Digitized by Google

CONTENTS.

Mathematical Papers, &c.

Calcul des côtés (1) du triangle et hexagone réguliers; (2) des décagones. (Professor L. Catalá) 40

Note on the Solution of certain Equations. (A. M. Nesbitt, M.A.) 41

Remarks on Mr. A. M. Nesbitt's Note on the Solution of certain Equations. (Professor Sanjána, M.A.) 69

Geometrical Note on Inversion. (R. F. Davis, M.A.)..... 71

Algebraical Note. (A. M. Nesbitt, M.A.) 93

Solved Questions.

830. (M. Collins, B.A.)—From a point in a line of the third order that has a double point, only two tangents can be drawn to the curve; prove that the lines drawn from the double point to the points of contact together with the two tangents at the double point form a harmonic pencil 28

1507. (The late Professor Clifford, F.R.S.) — Consider six planes A, B, C, D, E, F and join the point ABC to the point DEF, and so on; we have thus ten finite straight lines and their middle points lie in a plane 59

4950. (Professor Clifford, F.R.S.)—Prove that every matrix of the second order may be expressed in the form $aI + bJ$, where I is the matrix unity and J a matrix such that $J^2 = -I$. Hence find an expression for any power of such a matrix. (See Cayley on "Matrices," *Phil. Trans.*, 1858.) Required a geometrical representation for a non-self-conjugate linear and vector function 108

6487. (Rev. Preb. W. A. Whitworth, M.A.) — Express all the numbers from 1 to 140 by four nines. Algebraical symbols and decimal points may be used, but the expression for each number must contain the four nines *and no other figure*. Also express the same series of numbers by four fours 43

8059. (T. Muir, LL.D., F.R.S.E.)—Prove that the Pfaffian

$$\begin{vmatrix} a_1+a_2, & a_1+a_3, & a_1+a_4, & \dots, & a_1+a_{2n} \\ & a_2+a_3, & a_2+a_4, & \dots, & a_2+a_{2n} \\ & & a_3+a_4, & \dots, & a_3+a_{2n} \\ & & \dots & \dots & \dots \\ & & & & a_{2n-1}+a_{2n} \end{vmatrix} = 2^{n-1}(a_1a_3a_5 \dots a_{2n-1} + a_2a_4a_6 \dots a_{2n})$$

..... 97

8735. (A. Gordon.)—If

$$(x^2/a^2 + y^2/b^2 + z^2/c^2 - 1)\theta + (x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2 = 0$$

represent a cone, prove that $\alpha^2 + \theta$ is one of the primary axes of the confocals through (α, β, γ) 42

Digitized by Google

9308. (Professor Nilkantha Sarkar, M.A.)—A uniform rod rests in a horizontal position, being supported by two smooth pegs equidistant from its centre. If the distance between the pegs $= 1/\sqrt{3}$ times the length of the rod, prove that, if one of the pegs be suddenly removed, the pressure on the other will be initially unaltered 99

9388. (Professor Nash, M.A.)—Show that (1) the foci of the conic $l\alpha^2 + m\beta^2 + n\gamma^2 = 0$ are given by the equations $l\alpha A = m\beta B = n\gamma C$, where $A = \beta\gamma + a(\beta \cos B + \gamma \cos C - a \cos A)$, ...; and hence (2), if the conic touch four fixed straight lines, the locus of the foci is a circular cubic passing through the six vertices of the quadrilateral formed by the lines and also through the feet of the perpendiculars of the harmonic triangle of the quadrilateral; and (3), if the conic pass through four fixed points, the locus of the foci is a bicircular sextic having double points at the vertices (acnodes) and at the feet of the perpendiculars (crunodes) of the harmonic triangle of the quadrangle 17

9537. (Professor Plamenewski, M.A.)—The rate of interest increases gradually, the rate of increase at any time varying as the rate of interest. Interest being paid every instant, the principal and the rate of interest are both doubled in p years. Find the initial rate per cent. per annum 26

9700. (Professor Hanumanta Rau.)—Prove that, if n be even,

$$\sec^2 a + \sec^2(a + \pi/n) + \sec^2(a + 2\pi/n) + \ldots \text{ to } 2n \text{ terms} = 2n^2 \operatorname{cosec}^2 na$$

..... 41

9811. If masses P, Q, and R be placed at the vertices A, B, and C respectively of the triangle of reference, show that the trilinear equations to the principal axes at the centre of gravity of the masses will be

$$x + my + nz = 0 \quad \text{and} \quad l'x + m'y + n'z = 0,$$

where $l : m : n$ and $l' : m' : n'$ are determined by the equations

$$P(l/a) + Q(m/b) + R(n/c) = 0, \quad P(l'/a) + Q(m'/b) + R(n'/c) = 0,$$
$$P(ll'/a^2) + Q(mm'/b^2) + R(nn'/c^2) = 0,$$

and

$$ll' + mm' + nn' - (mn' + m'n)\cos A - (nl' + n'l)\cos B - (lm' + l'm)\cos C = 0,$$

and that at any point the principal axes are conjugate (*i.e.*, each passes through the pole of the other) with respect to the conics

$$Pl^2/a^2 + Qm^2/b^2 + Rn^2/c^2 = 0$$

and $$l^2 + m^2 + n^2 - 2mn\cos A - 2nl\cos B - 2lm\cos C = 0$$

(the circular points at infinity) 56

9870. (R. Tucker, M.A.)—O, Ω, Ω' are the in-centre and Brocard points of a triangle ABC. Ω_1, Ω_1'; Ω_2, Ω_2'; Ω_3, Ω_3' are the images of Ω, Ω' with respect to AO, BO, CO. Prove that, if ρ_1, ρ_2 are the radii of the circles round Ω, Ω_1, Ω_2, Ω_3; Ω', Ω_1', Ω_2', Ω_3' respectively, and if $\lambda^2 \equiv a^2b^2 + b^2c^2 + c^2a^2$, then

$$\rho_1^2 \sim \rho_2^2 = [abc\,\Pi(a-b)]/\lambda^2, \quad \rho_1^2 + \rho_2^2 = abc\,[\Sigma a\,\Sigma a^3 - (\Sigma a^2)^2]/(2s\lambda^2).$$

The equations to the two circles in trilinear co-ordinates are easily obtained from the fact that they are concentric with the in-circle of ABC 95

10173. (Rev. T. C. Simmons, M.A.)—Within a given closed convex area a random point is taken, and joined with any point on the perimeter. Show that, if two more random points be now taken in the area, the odds are two to one in favour of their falling on the same side of the line thus drawn 42

10349. (Professor Hain.)—Etant donné un triangle équilatéral ABC, si M, N, P sont les symétriques d'un point quelconque L par rapport aux trois côtés, démontrer que les droites AM, BN, CP se recontrent en un même point 59, 112

Digitized by Google

10390. (E. Césaro.)—Démontrer que la chaînette d'égale résistance, qui a pour équation intrinsèque $\rho = \frac{1}{2}a\,(e^{s/a} + e^{-s/a})$, est la seule ligne dont le rayon de courbure soit égal au segment de la normale intercepté par deux droites fixes parallèles 57

10582. (R. Knowles, B.A.) — The equation to a conic referred to a tangent and normal as axes being $ax^2 + bxy + cy^2 + gy = 0$, prove that, e being the eccentricity, the equation to its transverse axis is

$$\{4a\,(a+c) - (2-e^2)\,(b^2 - 4ac)\}\,x + 2b\,(a+c)\,y + bg\,(2-e^2) = 0 \quad \ldots \quad 72$$

11011. (Professor Crofton, F.R.S.) — Prove that the mean distances of the vertex of a triangle from all points in the area is equal to its distance from the centroid, *measured along a parabolic path*, which leaves the vertex in the direction of one of the sides, and reaches the centroid in a direction parallel to the others, the axis of the parabola being parallel to the base 32

11375. (Professor Zerr.)—Find (i.) the maximum rectangle, (ii.) the maximum parallelopiped, inscribed in, first, the curve $(x/a)^{\frac{1}{2}} + (y/b)^{\frac{1}{2}} = 1$ second, the surface $(x/a)^{\frac{1}{2}} + (y/b)^{\frac{1}{2}} + (z/c)^{\frac{1}{2}} = 1$ $(a > b > c)$ 75

12211. (Rev. T. R. Terry, M.A.) — If a and b are any positive quantities, find the sum of all the fractions $ab/[b + s\,(a+b)]$, where s may be zero or any integer, positive or negative 48

12229. (Professor Lévy.)—Un régiment marche à la cadence de 120 pas par minute; on demande quels sont les soldats qui poseront le pied droit sur le sol en même temps que la musique. Même problème en changeant la cadence 94

12264. (Professor Neuberg.) — Une droite m coupe les côtés d'un triangle ABC en trois points A″, B″, C″ tels que les perpendiculaires élevées en ces points sur BC, CA, AB forment un triangle A′B′C′ de surface donné. Trouver l'enveloppe de la droite m 49

12302. (Professor Clifford, F.R.S.) — C is the double point of a circular cubic, and a straight line cuts the curve in D, E, F; join CD, CE, CF, and on the two latter take A, B, so that CA . CE = CB . CF; then AB and CD are equally inclined to the tangent at C 30

12518. (Professor Russo.)—On donne, sur un même plan, une droite Δ et un cercle Γ fixes; soit Γ' un second cercle mobile, de rayon invariable, tangent à Δ. Trouver (1) l'enveloppe de l'axe radical δ des circonférences Γ et Γ'; et (2) le lieu du point de rencontre de δ avec la ligne des centres des circonférences, Γ, Γ'..... 76

12865. (W. Booth.) — If a plane be drawn touching the sphere $x^2 + y^2 + z^2 = R^2$ along its intersection with $z^2 = a^2x^2 + b^2y^2$, prove that the equation of the developable so generated is obtained by equating to zero the discriminant of $x^2/(1-\lambda a^2) + y^2/(1-\lambda b^2) + z^2/(1+\lambda) - R^2 = 0$ 105

12906. (Professor Orchard, B.Sc., M.A.)—Show from first principles that the time of a small oscillation of a hemisphere on any plane must be proportional to the square root of the radius 94

14625. (Professor Nanson.)—Tangents are drawn from $x'y'z'$, $x''y''z''$ to conics touching the lines

$$x = 0, \quad y = 0, \quad z = 0, \quad lx + my + nz = 0;$$

show that the locus of their intersection is

$$(lx + my + nz)\,[l\,(x'x''/x) + m\,(y'y''/y) + n\,(z'z''/z)] = (lx' + my' + nz')(lx'' + my'' + nz'') \quad \ldots \quad 63$$

Digitized by Google

14735. (Professor Morley.) — Given five planes a_1, ..., a_5, and a quadric Q, let a quadric Q_n pass through the intersection of Q and a_n ($n = 1, \ldots, 5$) and the four intersections of the four other planes. Then Q_n intersects Q in a second plane β_n, whose polar as to Q is a point a_n. Prove that the line a_1a_2 passes through the point (a_3, a_4, a_5); or, reciprocally, that the line a_1a_2 lies in the plane $a_3a_4a_5$. Also prove that, if one of the points a lies on Q, then all the points a lie on Q. Arrange the proof so as to hold for the analogous theorem in n dimensions 34

14754. (Rev. J. Cullen.) — Prove that the six sides of Tucker's triangles touch a conic the locus of whose centre is a cubic passing through the symmedian point and the centroid, having a double point where it cuts Euler's line again, and touching the S-point axis at the centroid 24

15014. (Professor Nanson.)—A conic touches BC, CA, AB in D, E, F. If DO, EO, FO cut the conic again in P, Q, R, prove AP, BQ, CR concurrent 76

15023. (R. Chartres.) — If C_r denote the combinations of n things r together, and D_r those of $(2n-1)$ things r together, sum the series

$$\frac{C_1C_1}{D_1} - \frac{C_2C_2}{D_2} + \frac{C_3C_3}{D_3} - \frac{C_4C_4}{D_4} + \ldots \quad \ldots\ldots\ldots\ldots 29$$

15080. (Professor Nanson.)—Lines drawn through the vertices of a triangle Δ divide the angles into the segments α, α'; β, β'; γ, γ', and form a triangle Δ'. If R, R' are the radii of the circum-circles of Δ, Δ', prove that

$$\frac{R'}{R} = \frac{\sin\alpha \sin\beta \sin\gamma - \sin\alpha' \sin\beta' \sin\gamma'}{\sin(\beta+\gamma')\sin(\gamma+\alpha')\sin(\alpha+\beta')},$$

and deduce the theorems of Ceva and Menelaus 57

15102. (J. Prescott, B.A.) — Given a square ABCD; draw a line through A to meet CD in X and BC produced in Y so that XY may be equal to a given length..... 65

15115. (R. Tucker, M.A.)—ABCD is a cyclic quadrilateral; (O_a, r_a), (O_b, r_b), (O_c, r_c), (O_d, r_d) are the in-centres and the in-radii of the triangles BCD, CDA, DAB, ABC respectively. Prove that $O_aO_bO_cO_d$ is a rectangle and $r_a + r_c = r_b + r_d$. Find also $\Sigma(OO_a^2)$ where O is the circum-centre 25

15142. (Profe-sor Nanson.)—If $x+y+z+w=0$, $ax+by+cz+dw=0$, $(a-d)^2(b-c)^2(xw+yz) + (b-d)^2(c-a)^2(yw+zx) + (c-d)^2(a-b)^2(zw+xy) = 0$,

prove that

$$x : y : z : w = (d-b)(d-c)(b-c) : (d-c)(d-a)(c-a) : (d-a)(d-b)(a-b) : (b-c)(c-a)(a-b)$$

..... 107

15198. (Professor Sanjána, M.A.)—ACA', BCB' are two fixed conjugate diameters of a hyperbola, and CP. CD are two conjugate semi-diameters: if AP, BD meet in O, and A'P, B'D in Q, prove that PODQ is a parallelogram, whose diagonal OQ is of fixed length and lies along an asymptote. Find also the maximum area of this parallelogram... 93

15216. (R. C. Archibald, M.A., Ph.D.)—The *trisectrix* of Maclaurin ($r\cos\frac{1}{3}\phi = C$ or $r\sin 2\theta + a\sin 3\theta = 0$), the *logocyclic* curve or *strophoide* ($r\cos\theta + a\cos 2\theta = 0$), and the *cubic* of Tschirnhausen ($r^{\frac{1}{3}}\cos\frac{1}{3}\theta = a^{\frac{1}{3}}$) are nodal cubics. Express their equations in the form $X^{\frac{1}{3}} + Y^{\frac{1}{3}} + Z^{\frac{1}{3}} = 0$, where X, Y, Z are linear functions of the Cartesian co-ordinates..... 53

Digitized by Google

15276. (θ. β.)—Let x, y be circular co-ordinates, and let the equations to three lines be $xt_s + y = x_s t_s$ ($s = 1, 2, 3$), where $|t_s| = 1$ and $x = x_s$, $y = x_s t_s$ is the reflection of the origin in the line. Verify that, for the triangle formed of these lines, the centroid is given by $x = a_1 - \frac{1}{3}a_2 p$, the orthocentre by $x = a_1 - a_2 p$, and the Simson-line of the point $x = a_1 - a_2 t$ by $$xt^3 - yt = -\tfrac{1}{3}a_2 t^3 + (a_1 - \tfrac{1}{3}a_2 p)t^2 - (b_1 - \tfrac{1}{3}a_2 q)\,t + a_2 r,$$
where $$a_1 = \Sigma x_1 t_1^2/(t_1 - t_2)(t_1 - t_3), \quad a_2 = \Sigma x_1 t_1/(t_1 - t_2)(t_1 - t_3),$$
$$b_1 = r\Sigma x_1/(t_1 - t_2)(t_1 - t_3),$$
$$p = t_1 + t_2 + t_3, \quad q = t_2 t_3 + t_3 t_1 + t_1 t_2, \quad r = t_1 t_2 t_3.$$
[For notation and methods, see Morley, *Transactions of the American Mathematical Society*, Vol. I., p. 97.] From this expression for the Simson-line we obtain immediately the well known result that its envelope is a hypocycloid of class 3, with the nine-point circle as its vertex-circle 36

15368. (Professor Nanson.)—If lines drawn through the vertices of a triangle Δ divide the angles into segments α, α'; β, β'; γ, γ', and form a triangle of area Δ', then
$$\frac{\Delta'}{\Delta} = \frac{(\sin\alpha \sin\beta \sin\gamma - \sin\alpha' \sin\beta' \sin\gamma')^2}{\sin(\beta+\gamma')\sin(\gamma+\alpha')\sin(\alpha+\beta')\sin(\alpha+\alpha')\sin(\beta+\beta')\sin(\gamma+\gamma')}$$
..... 57

15371. (Professor Neuberg.)—Trouver le lieu des centres des hyperboles équilatères qui passent par deux points donnés A, B et dont une asymptote passe par un troisième point donné C 100

15375. (Professor Nanson.)—Prove that
$$(a_1 X_2 Y_3 Z_4)(f_1 X_2 Y_3 Z_4) + b_1 (X_2 Y_3 Z_4)(g_1 X_2 Y_3 Z_4) + (c_1 X_2 Y_3 Z_4)(h_1 X_2 Y_3 Z_4) = 0$$
where $X_p, Y_p, Z_p, = h_p y - g_p z + a_p w$; $f_p z - h_p x + b_p w$; $g_p x - f_p y + c_p w$ 38

15393. (Professor Sanjána, M.A.)—In the series
$$S \equiv u_0 + u_1 + u_2 + u_3 + u_4 + u_5 + \ldots,$$
$u_0 = 1$, $u_1 = p$; also $u_2 = au_1 - u_0$, $u_3 = bu_2 - u_1$, $u_4 = au_3 - u_2$, $u_5 = bu_4 - u_3$, ... Prove that when S is convergent its sum
$$= [2ap - ab - \sqrt{(a^2b^2 - 4ab)}]/2(ap^2 - apb + b) \ldots\ldots 63$$

15400. (D. Biddle.) — AB, BC are two given straight lines which meet at right angles in B. Find in BC a point P such that, if BQ be drawn parallel to AP and CQ be perpendicular to it, CQ = 2BP..... 74

15407. ("Stranger.")—Find a point on a given parabola which is the centre of a circle touching the curve and passing through its focus 86

15438. (G. Heppel, M.A.)—P is any point within a circle. APA', BPB', CPC' are three chords making angles of 120° with one another. Prove that AP + BP + CP = A'P + B'P + C'P, and that each ranges between 3 and $\sqrt{3}$ times the radius 91, 102

15441. (Professor Nanson.)—Three triangles Δ_1, Δ_2, Δ_3 are inscribed in a conic. If Δ_2, Δ_3 are co-polar with Δ_1, then so is the triangle formed by the joins of corresponding vertices of Δ_2, Δ_3..... 76

15457. (Rev. T. Roach, M.A.)—Snow that, if the difference of two angles of a triangle is $\frac{1}{3}\pi$, the centre of the nine-point circle lies on the side adjacent to the two angles 20

15471. (Rev. T. Roach, M.A.)—Prove
(1) $\cosh^{-1}(\sec x) = \log\cot(\frac{1}{4}\pi - \frac{1}{2}x)$ or $\log\tan(\frac{1}{4}\pi - \frac{1}{2}x)$;
(2) $\sinh^{-1}(\tan x) = \log\cot(\frac{1}{4}\pi - \frac{1}{2}x)$ or $\log\tan(\frac{1}{4}\pi + \frac{1}{2}x)$ 98

Digitized by Google

15476. (A. M. Nesbitt, M.A.) — Find the linear factors of $abcfgh)(xyz)^2$ under the condition $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$. Apply the method to the case $(b-c)(x^2+2yz)+(c-a)(y^2+2zx)+(a-b)(z^2+2xy)=0$ 95

15479. (H. G. Graves.) — If x be the area of one of the faces of a tetrahedron; A, B, C the dihedral angles over x; and

$$M = \sqrt{(1-\cos^2 A-\cos^2 B-\cos^2 C-2\cos A\cos B\cos C)},$$

show that A/M has the same value for all the solid angles..... 67

15483. (Professor Sanjána, M.A.)—Find in positive integers all the solutions of $13k+21l+29m+37n=300$..... 21

15496. (R. F. Davis, M.A.)—Prove that

$\sin\theta = 2\sin\frac{1}{2}\theta-(2\sin\frac{1}{2}\theta\,2\sin\frac{1}{2}\theta\,2\sin\frac{1}{2}\theta)+(2\sin\frac{1}{2}\theta\ldots 2\sin\frac{1}{3^n}\theta)-\ldots$ ad inf. 23

15499. (Lt.-Col. Allan Cunningham, R.E. — *Revised.*) — Let $N = 111\ldots11$ consist of q^n units in the scale of $r = mq+1$, q being *prime;* then $N = Mq^n$ 49

15500. (Professor E. B. Escott.) — Find two large consecutive numbers ($>10^{14}$) all of whose prime factors are less than 100.

Examples: $a^2 = 7534241^2 = 11^2\cdot 13^2\cdot 19^2\cdot 47^2\cdot 59^2$,

$a^2-1 = 2^6\cdot 3^3\cdot 5\cdot 7^2\cdot 31^2\cdot 41^2\cdot 83$;

$a^2 = 15473809^2 = 13^4\cdot 19^2\cdot 61^2\cdot 79^2$,

$a^2-1 = 2^5\cdot 3^3\cdot 5\cdot 7^2\cdot 11\cdot 17\cdot 41\cdot 43\cdot 47\cdot 73$ 81

15501. (Professor Nanson.)—If $a>b>1$, and n is a positive integer, prove that $\frac{1}{a}\frac{a^{n+1}-1}{a^{n-1}-1} > \frac{1}{b}\frac{b^{n-1}-1}{b^{n-1}-1}$ 30

15509. (C. M. Ross.)—Having given the base and the altitude of a triangle, and that one of the angles at the base is double the other, show how to construct the triangle 104

15511. (A. M. Nesbitt, M.A.)—If

$$\lambda \equiv a\alpha/l+b\beta/m+c\gamma/n, \quad \lambda' = a\alpha/l'+b\beta/m'+c\gamma/n'$$

be two straight lines intersecting at ω and cutting the sides of the triangle of reference in (a, b, c), (a', b', c'), prove (1) that the four circles Abc, Bca, Cab, ABC meet in a point O, whose co-ordinates are

$$[a/(m-n),\ b/(n-l),\ c/(l-m)];$$

(2) that the point P common to the four circles Abc, $Ab'c'$, $\omega bb'$, $\omega cc'$ has its co-ordinates given by

$$a\alpha/\{ll'\delta K/\Delta-[a^2(B+c)]/L\} = M\beta/Bb = N\gamma/Cc$$

where $\Delta = \begin{vmatrix} ll & mm' & nn' \\ l & m & n \\ l' & m' & n' \end{vmatrix}$, A, B, C are the minors of ll', mm', nn' in Δ,

$\delta = \begin{vmatrix} 1 & 1 & 1 \\ l & m & n \\ l' & m' & n' \end{vmatrix}$, $L = (m-n)(m'-n')$, $M = (n-l)(n'-l')$, $N = \ldots$, and

$K = Aa^2/L+Bb^2/M+Cc^2/N$; and (3) that the points O, O', P lie on the circle

$$(abc\Delta/LMN)(a\alpha+b\beta+c\gamma)(La/a+M\beta/b+N\gamma/c) = K(a\beta\gamma+b\gamma\alpha+ca\beta),$$

Digitized by Google

an equation whose symmetry shows that it passes through Q, R, the respective common points of the circles (B$c$$a$, B$c'$$a'$, $\omega cc'$, $\omega aa'$) and (Cab, C$a'b'$, $\omega aa'$, $\omega bb'$) 26

15512. (R. Tucker, M.A.)—PQ is a chord of a parabola, constant in direction. PR, QR are also chords drawn in a constant direction. Show that the locus of R is a parabola..... 69

15516. (Hon. G. R. Dick.)—The surface whose co-ordinates at any point are given by

$$x = \rho\cos\phi,\quad y = \rho\sin\phi,\quad z = b\phi + c\int[(\rho^2+b^2)/(\rho^2-c^2)]^{\frac{1}{2}}\,d\rho/\rho$$

is a minimum surface: exhibit them in Monge's form; also the surface

$$\sin z = \sinh x \sinh y$$..... 65

15517. (C. M. Ross.)—Eliminate x, y, z from the equations

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy = 0,$$
$$a'x^2+b'y^2+c'z^2+2f'yz+2g'zx+2h'xy = 0,$$
$$a''x^2+b''y^2+c''z^2+2f''yz+2g''zx+2h''xy = 0,$$

expressing the eliminant as the equation of a determinant to zero ... 40

15518. (James Blaikie, M.A.)—A pack of 52 cards is shuffled in the following way:—The 2nd card is placed over the 1st, the 3rd under the 1st, the 4th over the 2nd, the 5th under the 3rd, and so on alternately. Find after how many such shuffles the pack will have regained its original order. Solve a similar question for packs of 10, 24, 30, 32 cards, and establish any general principles 77

15519. (Professor Sanjána, M.A.)—Find the condition that

$$x^9+ax^7+bx^5+cx^3+d$$

may be divisible by $(x+h)^3$. Also determine p, q, r in order that

$$x^9+px^7+qx^5+rx^3+1$$

may be divisible by $(x-1)^3$ or $(x+1)^3$ 78

15521. (Lt.-Col. Allan Cunningham, R.E.)—If σN denote the sum of the sub-factors of N (including 1, but excluding N), obtain some solutions of $\sigma N = X^2$ (an *even* square) 68

15522. (D. Biddle.) — Prove that, if a number (N) terminate in 2, 3, 7, or 8, it cannot be divided without remainder in such a way as to make the difference between the quotient and divisor a multiple of 5. Also state the general rule for finding the impossible terminals of N, where the modulus given is other than 5 (as above) 29

15526. (Professor Neuberg.)—Soient M un point fixe et N un point mobile d'une hyperbole équilatère. Trouver le lieu du point d'intersection de la normale en N à l'hyperbole avec la perpendiculaire élevée en M sur la corde MN 112

15527 (with changed notation). (L. Isserlis.)—Two conics touch at O. The distances of their centres from the tangent at O are h, H and their radii of curvature at O are r, R. The diameter of the first through O meets it again in A and the tangent at A meets the second conic in P, Q. Prove that, if $1/h-1/H = 1/\sqrt{(rh)}$, then the area of the triangle OPQ is $4R^{\frac{1}{2}}r^{-\frac{1}{2}}h^{\frac{1}{2}}$..... 97

15528. (A. M. Nesbitt, M.A.) — A triangle ABC is circumscribed by a hyperbola whose asymptotes are parallel to the bisectors of the angle A and its supplement. Find the angle at which the tangent at A cuts the opposite side..... 63

Digitized by Google

15536. (*Communicated by* J. L. S. Hatton.)—Evaluate the determinant

$$\begin{vmatrix} a & b & c & d & e & f & g & h \\ h & a & b & c & d & e & f & g \\ g & h & a & b & c & d & e & f \\ f & g & h & a & b & c & d & e \\ h & g & f & e & d & c & b & a \\ a & h & g & f & e & d & c & b \\ b & a & h & g & f & e & d & c \\ c & b & a & h & g & f & e & d \end{vmatrix}$$ 55

15537. (*Communicated by* H. J. Woodall, A.R.C.S.)—Interpret $x^{\sqrt{2}}$ 28

15539. (Lt.-Col. Allan Cunningham, R.E.)—For what forms of m is $X^3 \pm 1 \equiv 0 \pmod{m}$? Show how to find X when m is given. Ex.—$m = 99991$ 38

15541. (Rev. T. Roach, M.A.)—Sum the series

(1) $\frac{1}{1.2} - \frac{1}{5.6} + \frac{1}{9.10} - \ldots$, (2) $\frac{1}{2} - \frac{2}{5} + \frac{3}{8} - \frac{4}{11} + \ldots$ ... 22

15542. (D. Biddle.)—A triangle is inscribed in a circle of given radius, and has its centroid at a given distance from the centre. (1) Define the limits of area of inscribed triangles whose centroids are thus placed; and (2), given any area within those limits, inscribe a second triangle having this area, and whose centroid occupies the same position as that of the former or one equidistant from the centre of the circle ... 33

15543. (Professor S. Sircom.)—If D, E, F be any points on the sides BC, CA, AB respectively of a triangle ABC, and X, Y, Z the points in which EF, FD, DE respectively meet the same sides, then

$(DZ \cdot EX \cdot FY)/(DY \cdot EZ \cdot FX) = -(AF \cdot BD \cdot CE)/(AE \cdot BF \cdot CD)$ 90

15546. (Professor Nanson.)—Find the locus of the foci of a conic touching four given lines, and deduce the locus of the focus of a parabola touching three given lines 31, 63

15547. (Professor Sanjána, M.A.)—A central conic, whose minor axis is of given length, touches two fixed straight lines; if one focus moves on a given straight line, prove that the other moves on a hyperbola having its asymptotes parallel to the fixed lines 52

15549. (A. M. Nesbitt, M.A.)—CP, CQ are conjugate semi-diameters of an ellipse. If the triangle CPQ be of constant area, prove that the envelope of PQ is a similar ellipse 67

15553. (Professor Nanson.)—Eliminate x, y, z from

$$ax^2 + by^2 + cz^2 = 0,$$

$$(b-c)\,lyz + (c-a)\,mzx + (a-b)\,nxy = 0, \quad lx + my + nz = 0$$ 50

Digitized by Google

15554. (J. J. Barniville, B.A.) — Having $u_n + u_{n+3} = 3u_{n+1} + 3u_{n+2}$, prove that

$$\left(\frac{5}{-1.7}\right)^2 - \left(\frac{13}{7.17}\right)^2 + \left(\frac{53}{17.73}\right)^2 - \left(\frac{193}{73.263}\right)^2 + \ldots = \frac{1}{2},$$

$$\left(\frac{5}{1.7}\right)^2 - \left(\frac{17}{7.23}\right)^2 + \left(\frac{65}{23.89}\right)^2 - \left(\frac{241}{89.329}\right)^2 + \ldots = \frac{1}{2},$$

$$\left(\frac{13}{-7.17}\right)^2 - \left(\frac{25}{17.31}\right)^2 + \left(\frac{109}{31.51}\right)^2 - \left(\frac{389}{151.529}\right)^2 + \ldots = \frac{1}{98},$$

$$\left(\frac{17}{-7.23}\right)^2 - \left(\frac{37}{23.47}\right)^2 + \left(\frac{151}{47.217}\right)^2 - \left(\frac{565}{217.769}\right)^2 + \ldots = \frac{1}{98} \ldots$$ 35

15555. (Robt. W. D. Christie.) — Let $p = a^2 + b^2$, $q = ac + bd$, $s = c^2 + d^2$, $ad - bc = 1$: prove $r^{2m} + 1 = ps$ and $r^{6m} + 1 = p(p - 2q + s)$; and more generally $(pt + r^m)^2 + 1 = pS$, $(pt + r^{3m})^2 + 1 = pT$, where r is any primitive root of prime $4m + 1$, t any integer. Establish similar equations for prime $p = 4m - 1$ 52

15556. (Lt.-Col. Allan Cunningham, R.E.) — Factorise completely into *prime* factors $N = G_4^4 + 8 + F_4^4$, where $G_n = F_0 . F_1 . F_2 . \ldots . F_{n-1}$ and $F_n = 2^{2^n} + 1$, a Fermat's number 72

15557. (James Blaikie, M.A.) — If n be a number of r digits such that its square ends in the same r digits as the number, prove that (*a*) $10^r + 1 - n$ has the same property; (*b*) the square of $n - 1$ ends in the same r digits as $10^r + 1 - n$; (*c*) the cube of $(n - 1)$ ends in the same r digits as $n - 1$. Also, if n be a number of r digits such that its cube ends in the same r digits, prove that (*d*) $10^r - n$ has the same property. Examples:

$890625^2 = \ldots 890625$, (*a*) $109376^2 = \ldots 109376$, (*b*) $890624^2 = \ldots 109376$, (*c*) $890624^3 = \ldots 890624$, $6249^3 = \ldots 6249$, (*d*) $3751^3 = \ldots 3751$ 60

15559. (Professor R. W. Genese, M.A.)—An arc PQ of a uniform wire in the shape of an equiangular spiral attracts a particle at the pole O according to the law of nature. Prove that the resultant attraction is proportional to chord PQ/(OP . OQ) 46

15563. (R. F. Davis, M.A.)—Given a hyperbola whose eccentricity = 2, prove that the circle described upon any focal chord as diameter passes through the further extremity of the transverse axis 37, 54

15565. (Hon. G. R. Dick.)—If the bisectors of the base angles B and C of a triangle ABC meet the opposite sides in E and F, BC being given and the angle A constant, the line EF will touch a conic whose eccentricity is $\frac{1}{2}$ cosec $\frac{1}{2}$A 61, 111

15567. (Solidus.) — Two equal circular discs are to be cut out of a rectangular piece of paper 10 inches long and 8 inches broad. What is the greatest possible diameter of the discs? 70

15571. (Professor Neuberg.) — Soient Ox, Oy, Oz trois axes rectangulaires. On prend sur Ox, Oy les longueurs $OA = a$, $OB = b$. Soit C un point quelconque de la droite AB. Trouver l'équation de la surface engendrée par une ellipse qui a pour demi-axes OC, OD, la droite OD étant prise sur l'axe OZ et égale à K . OC, K étant un nombre donné 61

Digitized by Google

15572. (Professor E. B. Escott.) — Find two large consecutive numbers ($>10^{17}$) all of whose prime factors are less than 200. Examples:

$$a^2 = 306 \cdot 331 \cdot 585^2 = 5^2 \cdot 7^3 \cdot 17^2 \cdot 19^2 \cdot 79^2,$$
$$a^2-1 = 2^7 \cdot 3 \cdot 11^2 \cdot 13 \cdot 31 \cdot 37 \cdot 83 \cdot 101 \cdot 107 \cdot 151;$$
$$a^2 = 315 \cdot 630 \cdot 756^2 = 17766^4 = 2^4 \cdot 3^{12} \cdot 7^4 \cdot 47^4,$$
$$a^2-1 = 5 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 37 \cdot 73 \cdot 89 \cdot 101 \cdot 109 \cdot 163;$$
$$a^2 = 940 \cdot 737006^2 = 2^2 \cdot 3^4 \cdot 11^4 \cdot 19^2 \cdot 127^2 \cdot 179^2,$$
$$a^2-1 = 5 \cdot 7 \cdot 13 \cdot 17 \cdot 23 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71 \cdot 103 \cdot 193 \ldots\ldots 81$$

15573. (Lt.-Col. Allan Cunningham, R.E.) — Factorise completely into *prime* factors $N = (208^{13}-1)$ 62

15574. (R. W. D. Christie.) — Find an infinite number of positive integral values of X *having the same* Y in the Pellian equation

$$X_n^2 - PY^2 = -1;$$

e.g., $70^2-29 \cdot 13^2 = -1$, $746^2-3293 \cdot 13^2 = -1$, ... 79

15575. (R. Chartres.) — If $mx-1 = r \cdot s$, $x>3$, then, if (1) r and s are consecutive integers, r the greater, $1/x$ will give six recurring (*bona fide*) figures in the scales $\left(nx^{+r}_{-s}\right)$, and three recurring (*bona fide*) figures in the scales $\left(nx^{+s}_{-r}\right)$; (2) but, if $r = s$, $1/x$ will give four such figures in the scales $(nx \pm r)$. Proof required without the modern theory of numbers 35

15576. (Solidus.)—Given

$$(c+x)(c-y) = (\mu^2-1)(1-c^2), \quad (z-x)(z+y) = (\mu^2-1)(1-z^2);$$

prove that the equations got by interchanging x, y, and μ with c, z, and $1/\mu$ must also hold good 47

15577. (Professor H. Langhorne Orchard, M.A., B.Sc.) — Find (1) the sum of n terms of the series

$$0 + 128 + 4374 + 49152 + 312500 + 1399680 + \ldots;$$

and (2) the sum of n terms of the series in (1) added to n terms of the series $1 + 128 + 2187 + 16384 + 78125 + 279936 + \ldots$ 58

15580. (Professor Sanjána, M.A.) — A parabola has contact of the second order with $y^2 = 4ax$ at the vertex, and cuts it again at the point whose normal is inclined at $\tan^{-1} m$ to the axis. Prove that its equation is $(4x+my)^2 = 4am^2x$..... 68

15581. (C. E. Youngman, M.A.)—If A, B, C, D, E be five points on a rectangular hyperbola, A, B, C, D concyclic, DD′ a diameter of the circle, EE′ of the hyperbola, then the conics ABCD′E and ABCD′E′ will be ellipse and hyperbola, with their axes parallel and proportional 73

15583. (Rev. T. Roach, M.A.)—Simplify

$$[(1-e^{i\theta}\cos\theta)/(1-e^{-i\theta}\cos\theta)]^{\frac{1}{2}}$$..... 110

15584. (James Blaikie, M.A.)—Let O be any point in the plane of a triangle ABC, and let any straight line through O meet BC in P, CA in Q, AB in R; then, if points P′, Q′, R′ be taken on the line, so that PO = OP′, QO = OQ′, RO = OR′, prove that AP′, BQ′, CR′ are concurrent 48

Digitized by Google

15586. (*Communicated by* Frank Porter, M.A.)—ABC is an isosceles triangle, base angles being 80°; ∠EBC = 50°, ∠BCD = 60°; DE is joined and produced to meet the base produced in F. Find the angle EFC by means of propositions in Euclid, Book I. 64

15589. (R. Chartres.) — Express $m^n/(mr-1)$ as a radix fraction in scale r 55

15594. (J. J. Barniville, B.A.)—Having

$$u_1 = 2b^2-a^2, \qquad u_2 = a^2+4ab+2b^2,$$
$$u_3 = a^2+8ab+14b^2, \qquad u_4 = 7a^2+36ab+46b^2, \quad \ldots,$$

and $v_1 = a^2+2ab+2b^2$, $v_2 = a^2+6ab+10b^2$, $v_3 = 5a^2+26ab+34b^2$, ..., the scale of each series being x^3-3x^2-3x+1, prove that $u_n^2+u_{n+1}^2 = 2v_n^2$ 91

15595. (Professor Sanjána, M.A.)—Prove that

$$\cosh \tfrac{1}{2}u \cosh \tfrac{1}{4}u \cosh \tfrac{1}{8}u \ldots \text{ ad inf.} = (\sinh u)/u.$$

Hence deduce the sums of the series, ad inf.,

(1) $\dfrac{x}{1+x} + \tfrac{1}{2}\dfrac{x^2}{1+x^2} + \tfrac{1}{4}\dfrac{x^4}{1+x^4} + \ldots$, (2) $\dfrac{1}{1+x} + \tfrac{1}{2}\dfrac{1}{1+x^2} + \tfrac{1}{4}\dfrac{1}{1+x^4} + \ldots$ 74

15599. (A. M. Nesbitt, M.A.)—Two tangents drawn from a point T to a parabola meet any third tangent in F and F′. If the diameter through T meet the curve in P, prove that the tangent at P bisects FF′ 85

15600. (*Communicated by* Rev. T. Roach, M.A.)—Four pennies are placed flat on a table so that each touches two of the others. Find when the space enclosed between them is a minimum or maximum... 66

15602. (L. Isserlis.) — The corners *ab*, *bc*, *cd*, *da* of a quadrilateral whose sides are *a*, *b*, *c*, *d* are joined by lines *q*, *r*, *s*, *p* to the corners $\alpha\beta$, $\beta\gamma$, $\gamma\delta$, $\delta\alpha$ of a quadrilateral whose sides are α, β, γ, δ. a, α meet in pq, b, β in qr, c, γ in rs, and d, δ in sp. The lines rs, sp and qr, pq meet in pr; rp, rb, and a meet in bp; and rs, qr and sp, pq in qs. Prove that the lines qs, ba and ps, bp meet on c 107

15604. (Professor Lauvernay.) — Construire un triangle rectangle connaissant en position une droite Δ sur laquelle se trouve l'hypoténuse, le centre Ω du cercle enveloppant les cercles ex-inscrits au triangle ainsi que son rayon ρ 55

15605. (James Blaikie, M.A.)—A. and B. play at lawn-tennis, and A.'s chance of winning each point is to B.'s chance as m to n. Find A.'s chance of winning the game (1) when the score stands at "deuce," (2) when it stands at A.'s vantage, (3) at B.'s vantage 105

15606. (Rev. J. Cullen.)—Given four vectors α, β, γ, δ, show that, if $\phi(t) = \mathrm{V}(\alpha+t\beta)(\gamma+t\delta)^{-1}$, then $\mathrm{U}\Sigma\mathrm{V}\,\phi(t_2)\phi(t_3)$ is invariable for all values of t_1, t_2, t_3, provided $t_1 > t_2 > t_3$. Hence obtain the equation of the plane formed by the intersection of the reciprocal cone (whose vertex is the origin) with the cylindroid determined by two given screws (σ, ω) and (σ_1, ω_1) 92

15608. (Lt.-Col. Allan Cunningham, R.E.) — Factorize into *prime* factors $\mathrm{N} = (2^{64} \mp 2^{32}+1)^2+1$ 106

15626. (R. Chartres.) — If C_r, D_r be the combinations of n and $2n$ things respectively taken r together, find the sum of

$$1+(C_1)^2/D_2+(C_2)^2/D_4+(C_3)^2/D_6+\ldots \quad \ldots\ldots 87$$

Digitized by Google

15627. (A. H. Bell.)—Give a formula for all integral right-angled triangles where the difference between the sides containing the right angle is 7. Examples: 5, 12, 13; 8, 15, 17; 88

15628. (Robert W. D. Christie.)—If $x_1^2+1 = p_1 y^m$ for all integral values of m, then $x_n^2+1 = p_n y^m$ ad inf. Generalise.

E.g.: $239^2+1 = 2 \cdot 13^4$; then $85922^2+1 = 5 \cdot 17 \cdot 3041 \cdot 13^4$ ad inf. 103

15629. (A. W. T.)—A number is raised to the power (-1) and reduced to a decimal. It is found that the original number is just n times the digit nearest the decimal point. Find the limits between which the number lies 97

15630. (Professor Neuberg.)—Résoudre les systèmes

(1) $\sin x = a \cos y$, $\sin y = b \cos z$, $\sin z = c \cos x$;

(2) $\sin x = a \tan y$, $\sin y = b \tan z$, $\sin z = c \tan x$ 110

15638. (A. M. Nesbitt, M.A.)—The normals to a parabola at P and P′ meet the axis in G and G′. If H be the mid-point of GG′, prove geometrically that HP = HP′ 100

15644. (W. H. Blythe, M.A.)—If M divides the period of 1/M, or, in other words, if 1/M and $1/M^2$ have the same period, then $M = 2(3^n)+1$, where n is a positive integer, and a multiple of 5, including zero. Is this the case? It is to be observed that, being true for 3 and 487, it is true also for their product 1461 109

15645. (R. Chartres.)—The reciprocals of two consecutive numbers give the same number of recurring figures (n) in the scale radix (r), and the results looked upon as whole numbers are consecutive. Find the numbers and r and n..... 106

ERRATA.

Page 48. Unavoidably, the solutions to Questions 12211 and 15584 slightly anticipate their natural positions in the volume.

,, 81. In the statement of Question 15572 omit the periods in the values assigned to a^2.

Digitized by Google

MATHEMATICS

FROM

THE EDUCATIONAL TIMES,

WITH ADDITIONAL PAPERS AND SOLUTIONS.

9388. (Professor NASH, M.A.)—Show that (1) the foci of the conic $l\alpha^2 + m\beta^2 + n\gamma^2 = 0$ are given by the equations $l\alpha A = m\beta B = n\gamma C$, where $A = \beta\gamma + \alpha(\beta \cos B + \gamma \cos C - \alpha \cos A)$, ...; and hence (2), if the conic touch four fixed straight lines, the locus of the foci is a circular cubic passing through the six vertices of the quadrilateral formed by the lines and also through the feet of the perpendiculars of the harmonic triangle of the quadrilateral; and (3), if the conic pass through four fixed points, the locus of the foci is a bicircular sextic having double points at the vertices (acnodes) and at the feet of the perpendiculars (crunodes) of the harmonic triangle of the quadrangle.

[The foci can, of course, be obtained as the intersections of two conics, instead of two circular cubics, but apparently not in a symmetrical form.]

Solutions (I.) *by* H. BATEMAN, B.A.; (II.) *by* A. M. NESBITT, M.A., *and* Professor SANJÁNA, M.A.

(1) Let (α, β, γ), $(\alpha', \beta', \gamma')$ be the foci; the tangential equation of the conic is

$$(\lambda\alpha + \mu\beta + \nu\gamma)(\lambda\alpha' + \mu\beta' + \nu\gamma') + k\{\lambda^2 + \mu^2 + \nu^2 - 2\mu\nu \cos A - 2\nu\lambda \cos B - 2\nu\lambda \cos C\} = 0.$$

If this is of the form $\lambda^2/l + \mu^2/m + \nu^2/n = 0$,

$$\frac{\beta\gamma' + \beta'\gamma}{\cos A} + \frac{\gamma\alpha' + \gamma'\alpha}{\cos C} = \frac{\alpha\beta' + \alpha'\beta}{\cos C} = 2k = \frac{u\{\beta\gamma' + \beta'\gamma\} + v\{\gamma\alpha' + \gamma'\alpha\} + w\{\alpha\beta' + \alpha'\beta\}}{u \cos A + v \cos B + w \cos C};$$

i.e., the foci are conjugate points with regard to any rectangular hyperbola circumscribing the triangle, *i.e.*, they are isogonal conjugate points with regard to the pedal triangle (this can also be easily proved geometrically).

Digitized by Google

Let $\alpha = (g+h)\sec A, \quad \beta = (h+f)\sec B, \quad \dots;$

then $\alpha' = (1/g + 1/h)\sec A, \quad \beta' = (1/h + 1/f)\sec B, \dots;$

therefore $2k = \{[(g+h)(h+f)(f+g)]/fgh\}\sec A \sec B \sec C;$

therefore the tangential equation of the conic is

$$\lambda^2\{2f(g+h)^2\sec^2 A + (g+h)(h+f)(f+g)\sec A \sec B \sec C\} + \dots$$

or $$\lambda^2\{\alpha^2(\beta\cos B + \gamma\cos C - \alpha\cos A) + \alpha\beta\gamma\} + \dots;$$

therefore $l\alpha A = m\beta B = n\gamma C.$

(2) If the conic touch four fixed lines, we have $\lambda^2/l + \mu^2/m + \nu^2/n = 0;$ therefore the locus of the foci is the cubic

$$\lambda^2\{\alpha^2(\beta\cos B + \gamma\cos C - \alpha\cos A) + \alpha\beta\gamma\} + \dots.$$

The cubic must be its own isogonal transformation with regard to the pedal triangle; therefore it passes through the angular points of this triangle. Also the circular points are the imaginary foci of a parabola touching the four lines; therefore it is a circular cubic. The other six points are obviously on the cubic.

(3) If the conic passes through the four points $\pm f, \pm g, \pm h,$ we have $lf^2 + \dots + mg^2 + nh^2 = 0;$ therefore the locus of the foci is the sextic; therefore $$f^2/[\alpha^2(\beta\cos B + \gamma\cos C + \alpha\cos A) + \alpha\beta\gamma] + \dots = 0.$$

In order that this sextic should be its own isogonal transformation with regard to the pedal triangle it is necessary that it should have double points at the corners of this triangle. The sextic also has double points at the corners of the triangle of reference; for the equation is of the form

$$\alpha^2 P + \alpha\beta\gamma Q + \beta^2\gamma^2 R = 0.$$

If $\rho_1, \rho_2, \rho_3, \rho_4$ be the distances of the focus from the four points; $(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)$ their co-ordinates referred to fixed axes, we have

$$\rho_1 = Px_1 + Qy_1 + R, \quad \rho_2 = Px_2 + Qy_2 + R,$$
$$\rho_3 = Px_3 + Qy_3 + R, \quad \rho_4 = Px_4 + Qy_4 + R;$$

therefore, eliminating P, Q, R, we have an equation of the form

$$A\rho_1 + B\rho_2 + C\rho_3 + D\rho_4 = 0.$$

This represents a bicircular sextic having the four points as double foci.

If the four points are concyclic, the sextic breaks up into two cubics. In this case the axes of all conics through the four points are in fixed directions: accordingly, if we take the axes of reference parallel to them, we have either of the systems

$$\rho_1 = ex_1 + c, \quad \rho_2 = ex_2 + c, \quad \rho_3 = ex_3 + c, \quad \rho_4 = ex_4 + c;$$
$$\rho_1 = e'y_1 + d, \quad \rho_2 = e'y_2 + d, \quad \rho_3 = e'y_3 + d, \quad \rho_4 = e'y_4 + d;$$

therefore $\rho_1(x_2 - x_3) + \rho_2(x_3 - x_1) + \rho_3(x_1 - x_2) = 0,$ and similar equations. Therefore in this case the locus consists of the two circular cubics having the four points as foci.

(II.) The equation to the circular asymptotes through (p, q, r) is (see Note A):—

$$\alpha^2(q^2 + r^2 + 2qr\cos A) + \dots + \dots - 2\beta\gamma\{qr + p(q\cos B + r\cos C - p\cos A)\} - 2\gamma\alpha\{\dots\} - 2\alpha\beta\{\dots\} = 0,$$

Digitized by Google

or, say, $(P, Q, R, -L, -M, -N)(\alpha\beta\gamma)^2 = 0$ .....(1).

But, if (p, q, r) be a focus of

$$S \equiv l\alpha^2 + m\beta^2 + n\gamma^2 = 0 \quad \text{.....(2)},$$

(1) and (2) have double contact.

If, then, we take for their common chord $x\alpha + y\beta + z\gamma = 0$, we must have

$$P\alpha^2 + \ldots + \ldots - 2L\beta\gamma - \ldots - \ldots + (x\alpha + y\beta + z\gamma)^2 = 0$$

identical with S. Hence $yz = L$, $zx = M$, $xy = N$; whence $x^2 = MN/L$.

Also $PL + MN = l\theta$, $QM + NL = m\theta$, $RN + LM = n\theta$.

Now, it may be shown without difficulty that

$$PL + MN = qr(p \sin A + q \sin B + r \sin C)^2,$$

with similar expressions for the others; so that $pLl = qMm = rNn$ where (p, q, r) are the co-ordinates of a focus and L, M, N are the A, B, C of the Question. From this it follows that, if S touch the four lines $x\alpha \pm y\beta \pm z\gamma = 0$ (whose harmonic triangle is the triangle of reference), so that we have the identical relation $x^2/l + y^2/m + z^2/n = 0$, the foci will lie on the cubic $x^2L\alpha + y^2M\beta + z^2N\gamma = 0$. Now,

$L \sin A = \beta\gamma \sin A + \gamma\alpha \sin B + \alpha\beta \sin C - \alpha \cos A\,(\alpha \sin A + \beta \sin B + \gamma \sin C)$;

therefore L (and, similarly, M and N) must vanish at both focoids; and thus the cubic is circular. Moreover, since the substitution $\gamma = 0$, $x = \pm\beta$, $y = \alpha$ satisfies its equation identically, it passes through the six vertices, whose co-ordinates are $(0, z, \pm y)$, $(\pm z, 0, x)$, $(y, \pm x, 0)$. And, since L may be written $\gamma(\beta + \alpha \cos C) + \alpha(\beta \cos B - \alpha \cos A)$, we see that L and M vanish where $y = 0$ meets $\beta \cos B = \alpha \cos A$, *i.e.*, at the foot of the perpendicular to the side γ.

Again, if the four fixed points be $\alpha/x = \pm\beta/y = \pm\gamma/z$, so that the harmonic triangle of the quadrangle is the triangle of reference, we have at once $lx^2 + my^2 + nz^2 = 0$; so that the foci lie on the sextic

$$x^2MN\beta\gamma + \ldots + \ldots = 0,$$

which, since L, M, N all vanish at both focoids, must pass twice through each; while, since the equation may be written

$$N\gamma(x^2M\beta + y^2L\alpha) + z^2LM\alpha\beta = 0,$$

it also passes twice through the foot of the perpendicular to the side γ; for at this point we have γ, L, M, all zero. In fact, $L = 0$ is the circle described on the side α as diameter. At the vertex A of the triangle of reference we have $M = 0$, $N = 0$, $\beta = 0$, $\gamma = 0$; hence this is also a double point. To find its nature we have to divide by α^4 and then make α infinite. We obtain

$$\cos A\,(x^2\beta^2 + y^2\gamma^2) + \beta\gamma(y^2 + z^2) = 0.$$

Now $(y^2 + z^2)^2 > 4y^2z^2 > 4y^2z^2 \cos^2 A$; so that the roots of this equation are real. These points are therefore *crunodes* and not *acnodes*.

Note.—The nature of the double points at the feet of the perpendiculars is less easy to determine; if, however, we make α and δ (δ being $\beta \cos B - \gamma \cos C$) simultaneously vanish, and divide by β^4, we find a quadratic equation in α, δ in which the coefficients of α^2 and δ^2 are of opposite signs; so that the roots are real.

Digitized by Google

Note A.—The points of intersection of

$$a\alpha + b\beta + c\gamma = 0 \quad \text{and} \quad \alpha/a + \beta/b + \gamma/c = 0$$

are given by $b^2 + c^2 + bc(\beta/\gamma + c/\beta) = 0$ or $\beta/\gamma + \gamma/\beta + 2\cos A = 0$; whence $\beta/\gamma = -e^{\pm Bi}$ and $\alpha/\gamma = -e^{\mp Bi}$; so that the co-ordinates of the focoids are $(-e^{\pm Bi}, -e^{\mp Ai}, 1)$; and the circular asymptotes through (p, q, r) are

$$\begin{vmatrix} e^{\pm Bi}, & e^{\mp Ai}, & -1 \\ p, & q, & r \\ \alpha, & \beta, & \gamma \end{vmatrix} = 0.$$

Denoting by X, Y, Z the co-factors of the first row, this yields

$$X\cos B + Y\cos A - Z \pm i(X\sin B - Y\sin A) = 0,$$

whose product is $\Sigma X^2 - 2\Sigma YZ\cos A = 0$. Now

$$X^2 - 2YZ\cos A = (q\gamma - r\beta)^2 - 2\cos A\,(r\alpha - p\gamma)(p\beta - q\alpha)$$
$$= 2\alpha^2 qr\cos A + r^2\beta^2 + q^2\gamma^2 - 2\beta\gamma(qr - p^2\cos A) - 2p\cos A\,(q\gamma\alpha + r\alpha\beta).$$

Thus, the coefficient of α^2 is $q^2 + r^2 + 2qr\cos A$, and that of $-2\beta\gamma$ is $qr - p^2\cos A + pq\cos B + pr\cos C$; so that equation (1) is established.

15457. (Rev. T. Roach, M.A.)—Show that, if the difference of two angles of a triangle is $\frac{1}{2}\pi$, the centre of the nine-point circle lies on the side adjacent to the two angles.

[Solutions far too numerous to allow of more than a few being published.—Editor.]

Solutions (I.) *by* R. Tucker, M.A.; (II.) *by* Rev. W. Anderson, M.A.; (III.) *by* S. C. Bagchi, B.A.

(I.) Let O and H be the circum-centre and orthocentre; therefore

$$AH = a\cot A = 2Oa,$$

where a is mid-point of BC; but

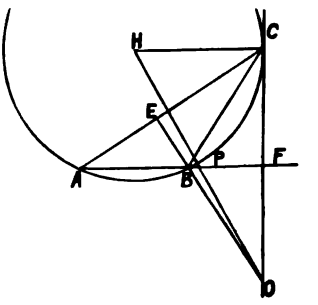
$a\cot A = a\tan 2B = 2b\sin A\cos B/\cos 2B = 2b\cos B = 2b\sin C = 2AD$; hence N lies on BD.

(II.) $ABC = \frac{1}{2}\pi + BAC$. H is centre of circum-circle. The perpendiculars CF and BE intersect at O. The angles OBF and CBF are each $90° - A$; therefore

$$CF = FO.$$

$$BCF = A;$$

therefore CF is the tangent to the circum-circle, and HC is parallel to AF; therefore the mid-point of OH, or nine-point centre of ABC, lies on AB.



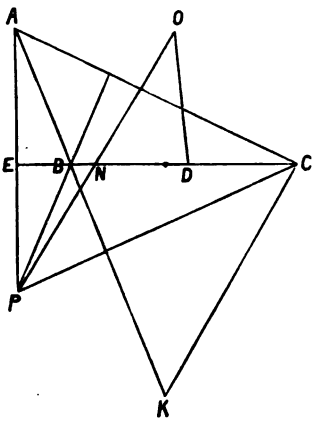
Digitized by Google

(III.) Let B > C. Draw CK perpendicular to AC to meet AB produced at K. Let P be the orthocentre, O the circum-centre. Join PO to cut BC in N. Then N will be the nine-point centre. Now B = 90° + C; therefore

$$\angle CBK = \angle BCK = 90° - C.$$

But CP is perpendicular to AK; therefore

$$\angle BCP = C = \angle ACB.$$



Also AP is perpendicular to BC produced; therefore E is the mid-point of AP. Since P is the ortho-centre and O the circum-centre, therefore AP = 2OD, where D is the mid-point of BC (by a well known relation between circum- and ortho-centres). But AP = 2PE (proved); therefore OD = PE, *i.e.*, $\triangle ODN \equiv \triangle PEN$; therefore PN = NO; therefore N is the nine-point centre (by a known theorem).

15483. (Professor Sanjána, M.A.)—Find in positive integers all the solutions of $13k + 21l + 29m + 37n = 300$.

Solutions (I.) *by* James Blaikie, M.A., *and* Frank Porter, M.A.; (II.) *by* Lt.-Col. Allan Cunningham, R.E., *and others.*

(I.) Since the coefficients of k, l, m, n are in arithmetic progression, it follows that, if a, b, c, d form one set of solutions, then other sets of solutions may be obtained in the following forms:—

$$a \pm r,\ b \mp r,\ c \mp r,\ d \pm r,\ \ldots,\quad a + r,\ b - 2r,\ c + r,\ d,\ \ldots \quad \ldots\ldots(1, 2),$$

$$a,\ b + r,\ c - 2r,\ d + r,\ \ldots \quad \ldots\ldots\ldots\ldots\ldots\ldots(3),$$

where r is any integer so long as all the expressions are positive.

The equation may be written in the form

$$13(k + l + m + n) + 8(l + 2m + 3n) = 300.$$

Let $k + l + m + n = 4x$, $l + 2m + 3n = y$, where it is evident that $12x > y$. We have now the equation $13x + 2y = 75$ to be solved in integers. It is evident that x is odd. If $x = 1$, $y = 31$, which is greater than $12x$, and therefore inapplicable. If $x = 3$, $y = 18$; and, if $x = 5$, $y = 5$. Taking the first of these, we get $k + l + m + n = 12$, $l + 2m + 3n = 18$, of which $k = l = m = n = 3$ is clearly a solution.

Applying (1), we get the solutions 0, 6, 6, 0; 1, 5, 5, 1; 2, 4, 4, 2; 3, 3, 3, 3; 4, 2, 2, 4; 5, 1, 1, 5; 6, 0, 0, 6.

Digitized by Google

Applying (2), we also get 1, 4, 7, 0 ; 2, 2, 8, 0 ; 3, 0, 9, 0 ; 2, 3, 6, 1 ; 3, 1, 7, 1 ; 3, 2, 5, 2 ; 4, 0, 6, 2 ; 4, 1, 4, 3 ; 5, 0, 3, 4.

Applying (3), we obtain the same sets of numbers in inverted order, viz. :—0, 7, 4, 1, ... There are thus twenty-five sets of solutions when $x = 3$. When $x = 5$, we obtain $k+l+m+n = 20$, $l+2m+3n = 5$, from which we deduce $k-m-2n = 15$, and one solution is clearly $k = 15$, $l = 5$, $m = 0$, $n = 0$.

Applying (2) and (3), we get 15, 5, 0, 0 ; 16, 3, 1, 0 ; 17, 1, 2, 0 ; 17, 2, 0, 1 ; 18, 0, 1, 1. There are thus thirty sets of solutions.

(II.) The equation may be written $13k+21l = 300-29m-37n$. Numerical values 0, 1, 2, ..., may be assigned to each of m, n, thus yielding a series of equations $13k+21l =$ a given number, containing only two unknowns (k, l), which may be easily solved. There appear to be 30 different solutions, as below. It will be seen that $k+l+m+n = 20$ or 12.

$k =$ 15, 3, 6, ., .	., 18, 17	., 4 5, 2	4	16, 17
$l =$ 5, ., ., 6, 9	7, ., 2	8, . ., 6	3	3, 1
$m =$., 9, ., 6, .	4, 1, .	2, 6 3, .	.	1, 2
$n =$., ., 6, ., 3	1, 1, 1	2, 2 4, 4	5	., .

$k =$ 1, 2	1, 2, 3	1, 2, 3	1, 2, 3, 4	3, 4	5
$l =$ 4, 2	5, 3, 1	6, 4, 2	7, 5, 3, 1	4, 2	1
$m =$ 7, 8	5, 6, 7	3, 4, 5	1, 2, 3, 4	1, 2	1
$n =$., .	1, 1, 1	2, 2, 2	3, 3, 3, 3	4, 4	5

15541. (Rev. T. Roach, M.A.)—Sum the series

(1) $\frac{1}{1.2}-\frac{1}{5.6}+\frac{1}{9.10}-\ldots$, (2) $\frac{1}{2}-\frac{2}{5}+\frac{3}{8}-\frac{4}{11}+\ldots$.

Solution by Professor Hudson, M.A.

(I.) (1) Write S for the sum ; then

$$S = 1-\frac{1}{2}-\frac{1}{5}+\frac{1}{6}+\frac{1}{9}-\frac{1}{10}-\ldots ;$$

therefore $4S = \frac{1}{\sqrt{i}}\log\frac{1+\sqrt{i}}{1-\sqrt{i}}+\frac{1}{\sqrt{(-i)}}\log\frac{1+\sqrt{(-i)}}{1-\sqrt{(-i)}}+i\log\frac{1+i}{1-i}$.

Digitized by Google

Expressing this free from i, we obtain

$$S = \tfrac{1}{4}[2\sqrt{2}\log(\sqrt{2}+1)+\pi(\sqrt{2}-1)] = \cdot 4742, \text{ approximately.}$$

(2) Write S for the series, and suppose that the number of terms is even; then $3S = \frac{1}{2}-\frac{1}{5}+\frac{1}{8}-\frac{1}{11}+\ldots$

$$= -\frac{1}{3}[\log(1+1)+\omega\log(1+\omega)+\omega^2\log(1+\omega^2)].$$

Expressing this free from ω, we obtain

$$S = \tfrac{1}{3\sqrt{3}}(\pi\sqrt{3}-\log 8) = \cdot 1245, \text{ approximately.}$$

If the number of terms is odd, add ·3333... to this.

Professor SANJÁNA, M.A., and Mr. C. M. Ross sum the first series in the following manner:—

Series (1)

$$= \frac{1}{1}-\frac{1}{2}-\frac{1}{5}+\frac{1}{6}+\frac{1}{9}-\frac{1}{10}-\frac{1}{13}+\frac{1}{14}\ldots$$

$$= \int_0^1 (1-x-x^4+x^5+x^8-x^9-x^{12}+\ldots)\,dx$$

$$= \int_0^1 [(1-x)-x^4(1-x)+x^8(1-x)-\ldots]\,dx = \int_0^1 \frac{(1-x)\,dx}{1+x^4}$$

$$= \int_0^1 \frac{dx}{1+x^4}-\int_0^1 \frac{x\,dx}{1+x^4} = \int_0^1 \frac{dx}{1+x^4}-\tfrac{1}{2}\int_0^1 \frac{dx}{1+x^2},$$

by putting $x^2 = x'$, and then suppressing accents. We have thus to evaluate $\left[\frac{1}{4\sqrt{2}}\log\frac{x^2+x\sqrt{2}+1}{x^2-x\sqrt{2}+1}+\frac{1}{2\sqrt{2}}\tan^{-1}\frac{x\sqrt{2}}{1-x^2}\right]_0^1-\frac{1}{2}\left[\tan^{-1}x\right]_0^1$; this gives

$$\frac{1}{4\sqrt{2}}\log\frac{2+\sqrt{2}}{2-\sqrt{2}}+\frac{1}{2\sqrt{2}}\,\frac{\pi}{2}-\frac{1}{2}\,\frac{\pi}{4} = \frac{1}{4\sqrt{2}}\log(\sqrt{2}+1)^2+\frac{\pi}{4}\left(\frac{1}{\sqrt{2}}-\frac{1}{2}\right).$$

$$= \frac{1}{4}\left\{\sqrt{2}\log(\sqrt{2}+1)+\frac{\pi}{2}(\sqrt{2}-1)\right\}.$$

15496. (R. F. DAVIS, M.A.)—Prove that

$$\sin\theta = 2\sin\tfrac{1}{2}\theta-(2\sin\tfrac{1}{2}\theta\,2\sin\tfrac{1}{4}\theta\,2\sin\tfrac{1}{8}\theta)+(2\sin\tfrac{1}{2}\theta\ldots 2\sin\tfrac{1}{2^3}\theta)-\ldots \text{ ad inf.}$$

Digitized by Google

Solutions (I.) *by* Rev. T. Wiggins, B.A.; (II.) *by* R. F. Whitehead, B.A., C. M. Ross, *and* Kela Deva Rau.

(I.) Call the terms of the dexter $u_1, u_2, \ldots$.

$$\sin\theta = 2\sin\tfrac{1}{2}\theta\,(1-2\sin^2\tfrac{1}{2}\theta) = u_1 - u_2\,(\cos\tfrac{1}{2}\theta) = u_1 - u_2 + u_2\,(2\sin^2\tfrac{1}{1\cdot 6}\theta)$$
$$= u_1 - u_2 + u_3\,(2\sin\tfrac{1}{1\cdot 6}\theta\, 2\sin\tfrac{1}{3\cdot 5}\theta\cos\tfrac{1}{3\cdot 5}\theta) = u_1 - u_2 + u_3\,(1-2\sin^2\tfrac{1}{5\cdot 7}\theta);$$

therefore
$$\sin\theta = u_1 - u_2 + u_3 - u_4 + \ldots.$$

(II.)
$$\sin\theta = 2\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta = 2\sin\tfrac{1}{2}\theta\,(1-2\sin^2\tfrac{1}{2}\theta)$$
$$= 2\sin\tfrac{1}{2}\theta - 2^3\sin\tfrac{1}{2}\theta\sin\tfrac{1}{4}\theta\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta$$
$$= 2\sin\tfrac{1}{2}\theta - 2^3\sin\tfrac{1}{2}\theta\sin\tfrac{1}{4}\theta\sin\tfrac{1}{2}\theta\,(1-2\sin^2\tfrac{1}{1\cdot 6}\theta)$$
$$= 2\sin\tfrac{1}{2}\theta - 2^3\sin\tfrac{1}{2}\theta\sin\tfrac{1}{4}\theta\sin\tfrac{1}{2}\theta + 2^5\sin\tfrac{1}{2}\theta\sin\tfrac{1}{4}\theta\sin\tfrac{1}{8}\theta\sin\tfrac{1}{1\cdot 6}\theta\sin\tfrac{1}{3\cdot 2}\theta - \ldots.$$

14754. (Rev. J. Cullen.)—Prove that the six sides of Tucker's triangles touch a conic the locus of whose centre is a cubic passing through the symmedian point and the centroid, having a double point where it cuts Euler's line again, and touching the S-point axis at the centroid.

Solution by G. N. Bates, M.A.

The co-ordinates of the angular points of D′E′F′ with respect to FDE are

D′ $-\sin\theta\sin(A-\theta)$, $\sin\theta\sin(B+\theta)$, $\sin(A-\theta)\sin(B+\theta)$;

E′ $\sin(B-\theta)\sin(C+\theta)$, $-\sin\theta\sin(B-\theta)$, $\sin C\sin(C+\theta)$;

F′ $\sin\theta\sin(A+\theta)$, $\sin(C-\theta)\sin(A+\theta)$, $-\sin\theta\sin(C-\theta)$.

The conic touching the six lines is

$$\sqrt{[a^2\sin(B+\theta)\sin(C-\theta)\,\alpha]} + \sqrt{[b^2\sin(C+\theta)\sin(A-\theta)\,\beta]} + \sqrt{[c^2\sin(A+\theta)\sin(B-\theta)\,\gamma]} = 0.$$

The "α" co-ordinate of its centre is proportional to

$$bc\,[c\sin(A+\theta)\sin(B-\theta) + b\sin(C+\theta)\sin(A-\theta)]$$

or
$$4R\sin B\sin C + (b\cos C - c\cos B)\,t - 2R\cos A\,.\,t^2 \quad (t = \tan\theta).$$

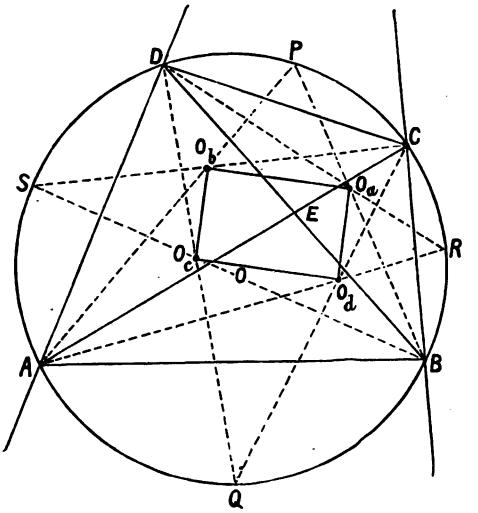
The "α" co-ordinate of the centre referred to ABC is, in terms of the old co-ordinates, $\alpha + [(a\cos B + \gamma)/\sin B]\,t$; therefore the locus of the centre is found by eliminating between

$$\alpha \,/\, [4R\sin B\sin C + 3at + 4R(\cos B\cos C - \cos A)\,t^2 - at^3] = \ldots = \ldots.$$

The locus is thus a cubic passing through G ($t = 0$), K ($t = \infty$), having a double point on the line OH ($t = \pm\sqrt{3}$), while its tangent at G is $\Sigma a\,(b^2 - c^2)\,\alpha = 0$ or GK.

Digitized by Google

15115. (R. Tucker, M.A.)—ABCD is a cyclic quadrilateral; (O_a, r_a), (O_b, r_b), (O_c, r_c), (O_d, r_d) are the in-centres and the in-radii of the tri-



angles BCD, CDA, DAB, ABC respectively. Prove that $O_aO_bO_cO_d$ is a rectangle and $r_a + r_c = r_b + r_d$. Find also $\Sigma(OO_a^2)$ where O is the circum-centre.

Solutions (I.) *by* J. H. Taylor, M.A., *and* J. Blaikie, M.A.;
(II.) *by the* Proposer *and others.*

(I.) Let the angles subtended by AB, BC, CD, DA at C, D, A, B respectively be 2α, 2β, 2γ, 2δ. Let Q, R, P, S be the middle points of the arcs AB, BC, CD, DA respectively. Then

$$\angle CDO_b = \alpha + \beta, \quad \angle PDO_b = \angle PDC + \angle CDO_b = \alpha + \beta + \gamma,$$

$$\angle DPO_b = \angle DBA = 2\delta; \quad \text{therefore } \angle PO_bD = \angle PDO_b;$$

therefore $$PO_b = PD = PC = PO_a.$$

Hence PQ bisects O_aO_b at right angles; therefore also QP bisects O_cO_d at right angles, and SR bisects O_cO_b and O_aO_d at right angles.

$$\angle SPQ = \angle SCQ = \alpha + \delta, \quad \angle PSR = \angle PAR = \gamma + \beta;$$

therefore PQ, SR intersect at right angles. Hence

$$O_aO_bO_cO_d \text{ is a rectangle} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

$\angle PAC = \gamma = \angle PBD$; therefore the parallels to AC and DB through O_b, O_a intersect, in p, on PQ. Hence $sprq$ is a rhombus, and its area

$$= (r_a + r_c)\,pr = (r_b + r_d)\,pq;$$

therefore $$r_a + r_c = r_b + r_d \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Digitized by Google

By a well known trigonometrical formula, in the triangle CDB, $OO_a^2 = R^2 - 2Rr_a$; therefore

$$\Sigma(OO_a^2) = 4R^2 - 4R(r_a + r_c) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

(II.) Since AO_cO_dB is cyclic, therefore $\angle AO_cO_d = \pi - \frac{1}{2}ABC$. Also $\angle AO_cO_b = \pi - \frac{1}{2}ADC$; therefore $\angle O_bO_cO_d = 2\pi - (2\pi - \frac{1}{2}\pi) = \frac{1}{2}\pi$; therefore, &c. Let $\angle ADB = \theta$, $\angle BDC = \theta'$, $\angle ABD = \phi$, $\angle DBC = \phi'$; then $r_a = 4R \sin\frac{1}{2}\phi' \sin\frac{1}{2}\theta' \sin\frac{1}{2}BCD$, $r_c = 4R \sin\frac{1}{2}\theta \sin\frac{1}{2}\phi \cos\frac{1}{2}BCD$; whence $r_a + r_c = R(\cos\theta + \cos\phi + \cos\theta' + \cos\phi' - 2) = r_b + r_d$; whence $\Sigma(OO_a)^2 = 4R^2[3 - (\cos\theta + \cos\phi + \cos\theta' + \cos\phi')]$.

9537. (Professor PLAMENEWSKI, M.A.)—The rate of interest increases gradually, the rate of increase at any time varying as the rate of interest. Interest being paid every instant, the principal and the rate of interest are both doubled in p years. Find the initial rate per cent. per annum.

Solution by CLEMENT V. DURELL.

Let r = rate of interest on £1 at time t (unit = 1 year),
k = constant of variation, P = principal at time t.

$dr/dt = kr$; therefore $r = Ae^{kt}$. If r_0 = initial rate of interest, *i.e.*, at time $t = 0$, $r_0 = A$; therefore $r = r_0e^{kt}$; also $dP = Prdt$; therefore $dP/P = r_0e^{kt}dt$; therefore $\log P = r_0e^{kt}/k + B$. If P_0 = principal at time $t = 0$, $\log P/P_0 = r_0(e^{kt} - 1)/k$; but $r = 2r_0$ and $P = 2P_0$, when $t = p$; therefore $e^{kp} = 2$ and $r_0(e^{kp} - 1)/k = \log 2$; therefore $k = (\log 2)/p$; therefore $r_0 = (\log 2)^2/p$; therefore initial rate of interest per cent. per annum $= 100(\log 2)^2/p$.

15511. (A. M. NESBITT, M.A.)—If

$$\lambda \equiv a\alpha/l + b\beta/m + c\gamma/n, \quad \lambda' = a\alpha/l' + b\beta/m' + c\gamma/n'$$

be two straight lines intersecting at ω and cutting the sides of the triangle of reference in (a, b, c), (a', b', c'), prove (1) that the four circles Abc, Bca, Cab, ABC meet in a point O, whose co-ordinates are

$$[a/(m-n),\ b/(n-l),\ c/(l-m)];$$

(2) that the point P common to the four circles Abc, $Ab'c'$, $\omega b\beta'$, $\omega c\gamma'$ has its co-ordinates given by

$$a\alpha/\{l'\delta K/\Delta - [a^2(B+c)]/L\} = M\beta/Bb = N\gamma/Cc$$

Digitized by Google

where $\Delta = \begin{vmatrix} ll' & mm' & nn' \\ l & m & n \\ l' & m' & n' \end{vmatrix}$, A, B, C are the minors of ll', mm', nn' in Δ, $\delta = \begin{vmatrix} 1 & 1 & 1 \\ l & m & n \\ l' & m' & n' \end{vmatrix}$; $L = (m-n)(m'-n')$, $M = (n-l)(n'-l')$, $N = \ldots$,—and $K = Aa^2/L + Bb^2/M + Cc^2/N$; and (3) that the points O, O′, P lie on the circle

$$(abc\Delta/LMN)(a\alpha + b\beta + c\gamma)(L\alpha/a + M\beta/b + N\gamma/c) = K(a\beta\gamma + b\gamma\alpha + c\alpha\beta),$$

an equation whose symmetry shows that it passes through Q, R, the respective common points of the circles $(Bca,\ Bc'a',\ \omega cc',\ \omega aa')$ and $(Cab,\ Ca'b',\ \omega aa',\ \omega bb')$.

Solution by the PROPOSER.

(1) It may be easily verified that the equation to the circle Abc is $bcIF = aS$, wherein $S \equiv \Sigma a\beta\gamma$, $I \equiv \Sigma aa$, and $F \equiv \beta/b\,(l-m) + \gamma/c\,(l-n)$; for this cuts $\beta = 0$ at the points given by $\gamma(lc\gamma + naa) = 0$; *i.e.*, at A and b, and so cuts $\gamma = 0$ at A and c. Thus $F = 0$ is the radical axis of ABC and Abc, and it obviously passes through the point

$$[a/(m-n),\ b/(n-l),\ c/(l-m)],$$

whose symmetrical co-ordinates prove that it also lies upon the circles Bca and Cab. This then is the point O.

(2) From this it follows that the radical axis AP of Abc and $Ab'c'$ has for its equation $lF = l'F'$, which reduces to $\beta M/(bB) = \gamma N/(cC)$. Now OO′ has for its equation $\Lambda \equiv \Sigma(L\alpha/a) = 0$; and any circle through O, O′—both of which points lie on S—may be denoted by $\theta I\Lambda = S$. Suppose now that the values of I, Λ, S, F when the co-ordinates of P are substituted for α, β, γ are $\bar{I}$, $\bar{\Lambda}$, $\bar{S}$, $\bar{F}$ respectively; the equation to the circle OO′P may then be written $\bar{S}I\Lambda = S\bar{I}\bar{\Lambda}$. Now, since P is on Abc, $\bar{S} = \frac{bcl}{a}\bar{I}\bar{F}$; and, if we take $\bar{\alpha}$, $\bar{\beta}$ $\left(\equiv \frac{bB}{M}\right)$, $\bar{\gamma}$ $\left(\equiv \frac{cC}{N}\right)$ for the co-ordinates of P, this yields $\bar{S} = \frac{bcll'\delta}{aMN}\bar{I}$; so that the equation to OO′P becomes $\frac{bcll'\delta}{aMN}I\Lambda = S\bar{\Lambda}$. By merely solving for $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$ between the equations of AP and the circle Abc, we find

$$\frac{\Delta a\bar{\alpha}}{ll'\delta(Bb^2/M + Cc^2/N) - a^2BC} = \frac{M\bar{\beta}}{Bb} = \frac{N\bar{\gamma}}{Cc} = 1,$$

and the first may be written $\frac{a\bar{\alpha}}{ll'\delta K/\Delta - [a^2(B+C)/L]}$; so that

$$\bar{\Lambda} = Lll'\delta K/a^2\Delta.$$

Finally, (3) the equation to the circle OO′P reduces to

$$abc\Delta/(LMN)I\Lambda = SK.$$

Note.—The foregoing proof by elementary methods of the third link in Clifford's chain may be of interest, and stimulate some competent mathematician to attempt a further step. I find without difficulty that

Digitized by Google

the radical centre of $p_1A_1I_1 = S$, $p_2A_2I_2 = S$, and $p_3A_3I_3 = S$, wherein $p_r = abc\Delta_r/(L_rM_rN_rK_r)$, has for co-ordinates

$$\frac{\alpha}{a\left(\frac{x_1}{m_1-n_1}+\frac{x_2}{m_2-n_2}+\frac{x_3}{m_3-n_3}\right)} = \frac{\beta}{b\left(\frac{x_1}{n_1-l_1}+\frac{x_2}{n_2-l_2}+\frac{x_3}{n_3-l_3}\right)} = \frac{\gamma}{c\left(\frac{x_1}{l_1-m_1}+\frac{x_2}{l_2-m_2}+\frac{x_3}{l_3-m_3}\right)} \left(x_r = \frac{\delta_r K_r}{\Delta r}\right),$$

but I have not come near to proving that this point lies, as it certainly does, upon any one of these circles.

---

830. (M. Collins, B.A.)—From a point in a line of the third order that has a double point, only two tangents can be drawn to the curve; prove that the lines drawn from the double point to the points of contact together with the two tangents at the double point form a harmonic pencil.

Additional Solution by θ. β.

If the tangent at the real inflexion is projected to infinity, and the line joining the double point to that inflexion is taken as $x = 0$, then (the double point being $x = 0$, $y = 0$) the cubic is reducible to the form $y^2 = x^2(x+a)$, *i.e.*, $x = t^2 - a$, $y = t^3 - at$, if $y/x = t$. Thus three points t_1, t_2, t_3 are collinear if $t_2t_3 + t_3t_1 + t_1t_2 + a = 0$. Take $t_3 = c$ as the point from which tangents are to be drawn; then, for a tangent, $t_1 = t_2 = t$, say, and so $t^2 + 2ct + a = 0$, giving two values of t only. The two roots of this quadratic form a harmonic set with those of $t^2 - a = 0$, which give the tangents at the node.

15537. (*Communicated by* H. J. Woodall, A.R.C.S.)—Interpret $x^{\sqrt{2}}$.

Solution by Lt.-Col. Allan Cunningham, R.E.

If $x^{\sqrt{2}} = N$, this admits of same interpretation as $x^z = N'$, where z is any incommensurable, viz., such that $\log_{10} N = \sqrt{2}\log_{10} x$; or, taking x itself as the base of a system of logarithms, then $\log_x N = \sqrt{2}$.

Note by the Proposer.

Let p/q, p'/q' be two consecutive convergents to $\sqrt{2}$; whence

$$pq' - p'q = \pm 1.$$

Then x [Ind. $\sqrt{2}$] will lie between x [Ind. p/q] and x [Ind. p'/q'].

If we consider only the real positive value of the expressions, we may write the indices in the forms pq'/qq', $p'q/qq'$; whence we may evaluate x [Ind. $\sqrt{2}$] as lying between the pq'-th and the $p'q$-th power of the qq'-th root of x; and the interpretation will follow the same lines.

Digitized by Google

15023. (R. Chartres.)—If C_r denote the combinations of n things r together, and D_r those of $(2n-1)$ things r together, sum the series

$$\frac{C_1C_1}{D_1}-\frac{C_2C_2}{D_2}+\frac{C_3C_3}{D_3}-\frac{C_4C_4}{D_4}+\dots.$$

Additional Solution by Solidus.

The series in question may be written

$$\text{(i.)}\quad \frac{n\,.\,n}{2n-1}\left[1-\frac{n-1}{2}\,\frac{(n-1)(n-2)}{(2n-2)(2n-3)}\right.$$
$$\left.\times\left\{1-\frac{n-2}{3}\,\frac{(n-3)(n-4)}{(2n-4)(2n-4)}\left(1-\frac{n-3}{4}\,\frac{(n-5)(n-6)}{(2n-6)(2n-7)}\dots\right)\right\}\right]\dots.$$

Its sum is $2n/(n+1)$, as may be proved by repeated applications of the identity

$$\text{(ii.)}\quad \frac{2r\,(2n-2r+1)}{n\,(n+1)}=1-\frac{n-r}{r+1}\,\frac{(n-2r+1)(n-2r)}{(2n-2r)(2n-2r-1)}\,\frac{2\,(r+1)(2n-2r-1)}{n\,(n+1)},$$

where r is put equal to 1, 2, 3, ... successively. Thus

$$\frac{2n}{n+1}=\frac{n\,.\,n}{2n-1}\,\frac{2\,(2n-1)}{n\,(n+1)}$$
$$=\frac{n\,.\,n}{2n-1}\left(1-\frac{n-1}{2}\,\frac{(n-1)(n-2)}{(2n-2)(2n-3)}\,\frac{4\,(2n-3)}{n\,(n+1)}\right)$$
$$=\frac{n\,.\,n}{2n-1}\left\{1-\frac{n-1}{2}\,\frac{(n-1)(n-2)}{(2n-2)(2n-3)}\right.$$
$$\left.\times\left(1-\frac{n-2}{3}\,\frac{(n-1)(n-4)}{(2n-4)(2n-5)}\,\frac{6\,(2n-5)}{n\,(n+1)}\right)\right\},$$

and so on.

Finally, clearing of brackets, we get

$$\frac{2n}{n+1}=\frac{n\,.\,n}{2n-1}-\frac{n\,(n-1)}{1\,.\,2}\,\frac{n\,(n-1)(n-2)}{(2n-1)(2n-2)(2n-3)}$$
$$+\frac{n\,(n-1)(n-2)}{1\,.\,2\,.\,3}\,\frac{n\,(n-1)(n-2)(n-3)(n-4)}{(2n-1)(2n-2)(2n-3)(2n-4)(2n-5)}-\dots,$$

which sums the proposed series.

15522. (D. Biddle.)—Prove that, if a number (N) terminate in 2, 3, 7, or 8, it cannot be divided without remainder in such a way as to make the difference between the quotient and divisor a multiple of 5. Also state the general rule for finding the impossible terminals of N, where the modulus given is other than 5 (as above).

Digitized by Google

Solution by Lt.-Col. ALLAN CUNNINGHAM, R.E.

This question can be readily solved in a general manner for any modulus (m). Let $N = X.Y$, and let ν, ξ, η be the least residues of N, X, Y on division by m, so that $N = mn + \nu$, $X = mx + \xi$, $Y = my + \eta$. Then, if one of X, Y be considered the divisor, the other (Y or X respectively) becomes the quotient.

Find first the condition that $(X \sim Y)$ may contain m; this gives $X \sim Y \equiv \xi \sim \eta \equiv 0 \pmod{m}$; whence $\xi = \eta$. And, since $N = XY$, hence $\nu \equiv \xi\eta \equiv \xi^2 \pmod{m}$. Thus, $(X \sim Y)$ is divisible by m whenever ν is a 2-ic residue of m; and is therefore not divisible by m when, and only when, ν is a 2-ic non-residue of m.

Ex.—Taking $m = 5$, its 2-ic non-residues are 2, 3; hence $N = 5n + 2, 3$; or $= 10n + 2, 3, 7, 8$ are the numbers required such that $X \sim Y$ does not contain 5.

12302. (Professor CLIFFORD, F.R.S.)—C is the double point of a circular cubic, and a straight line cuts the curve in D, E, F; join CD, CE, CF, and on the two latter take A, B, so that $CA \cdot CE = CB \cdot CF$; then AB and CD are equally inclined to the tangent at C.

Solution by W. J. GREENSTREET, M.A.

With C as pole, let D, E, F invert into D′, E′, F′. Draw tangents to the cubic at C; the inverse of the cubic (I) is a conic with asymptotes parallel to these tangents. DEF inverts into a circle CD′E′F′ which meets I. The lines CD, E′F′ are equally inclined to the above tangents, and, as AB and E′F′ are parallel, the theorem follows.

15501. (Professor NANSON.)—If $a > b > 1$, and n is a positive integer, prove that

$$\frac{1}{a}\,\frac{a^{n+1}-1}{a^{n-1}-1} > \frac{1}{b}\,\frac{b^{n-1}-1}{b^{n-1}-1}.$$

Solution by C. M. ROSS *and others.*

$$\frac{1}{a}\,\frac{a^{n+1}-1}{a^{n-1}-1} > \frac{1}{b}\,\frac{b^{n+1}-1}{b^{n-1}-1}$$

if $$b\,(a^{n+1}-1)\,(b^{n-1}-1) > a\,(a^{n-1}-1)\,(b^{n+1}-1)$$

if $$(a^n b^n - 1)\,(a-b) - (ab-1)\,(a^n - b^n) > 0$$

if $$(a-b)\,(ab-1)\left\{a^{n-1}b^{n-1} + a^{n-2}b^{n-2} + \ldots + 1 - (a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \ldots + b^{n-1})\right\} > 0 \quad (1).$$

Digitized by Google

Let $P \equiv a^{n-1}b^{n-1} + a^{n-2}b^{n-2} + \ldots + 1$, $Q \equiv a^{n-1} + a^{n-2}b + \ldots + b^{n-1}$. When n is odd the middle terms in P and Q are the $[\frac{1}{2}(n+1)]$-th (n denoting the number of terms); when n is even the middle terms in P and Q are the $\frac{1}{2}n$-th and $[\frac{1}{2}(n+1)]$-th.

The r-th term in P is $a^{n-r}b^{n-r}$, the r-th term in Q is $a^{n-r}b^{r-1}$.

Hence, when n is odd the $[\frac{1}{2}(n+1)]$-th term in P is $a^{\frac{1}{2}(n-1)}b^{\frac{1}{2}(n-1)}$, and when n is odd the $[\frac{1}{2}(n+1)]$-th term in Q is $a^{\frac{1}{2}(n-1)}b^{\frac{1}{2}(n-1)}$; therefore when n is odd there remain $(2n-2)$ terms in $P-Q$. Hence these terms can be grouped in fours; since $2n-2$ is exactly divisible by 4.

When n is even $2n$ is plainly divisible by 4.

Again,

$$(a-b)(ab-1)\{a^{n-1}b^{n-1} + a^{n-2}b^{n-2} + \ldots + 1 - (a^{n-1} + a^{n-2}b + \ldots + b^{n-1})\}$$
$$= (a-b)(ab-1)\{(a^{n-1}b^{n-1} - a^{n-1} - b^{n-1} + 1) + ab(a^{n-3}b^{n-3} - a^{n-3} - b^{n-3} + 1) + a^2b^2(a^{n-5}b^{n-5} - a^{n-5} - b^{n-5} + 1) + \ldots\}$$
$$= (a-b)(ab-1)\{(a^{n-1}-1)(b^{n-1}-1) + ab(a^{n-3}-1)(b^{n-3}-1) + a^2b^2(a^{n-5}-1)(b^{n-5}-1) + \ldots\}$$
$$= \text{a positive quantity (since } a > b > 1).$$

Therefore expression (1) > 0.

Therefore
$$\frac{1}{a}\,\frac{a^{n+1}-1}{a^{n-1}-1} > \frac{1}{b}\,\frac{b^{n+1}-1}{b^{n-1}-1}.$$

15546. (Professor Nanson.)—Find the locus of the foci of a conic touching four given lines, and deduce the locus of the focus of a parabola touching three given lines.

Solution by H. M. Taylor, M.A., F.R.S., *and* Professor Sanjána, M.A.

Let S, S′ be the foci of a conic which touches the sides of the triangle ABC and a fourth straight line DEF, cutting BC, CA, AB in D, E, F respectively. Let the trilinear co-ordinates of S, S′ be (x, y, z), (x', y', z') referred to ABC as the triangle of reference. Then, since the product of the perpendiculars from the foci of a conic on a tangent is constant, we have
$$xx' = yy' = zz' = [(lx + my + nz)/k][(lx' + my' + nz')/k]$$
where
$$k^2 = l^2 + m^2 + n^2 - 2mn\cos A - 2nl\cos B - 2lm\cos C.$$
As each of the above fractions is equal to $(lx' + my' + nz')/(l/x + m/y + n/z)$, we have
$$(lx + my + nz)(l/x + m/y + n/z) = k^2.$$

The locus of S, therefore, is a cubic passing through A, B, C, D, E, F, and touching the conic $l/x + m/y + n/z = 0$ at A, B, C. The tangent to the cubic at the point A is $ny + mz = 0$, which is the anti-parallel of AD with respect to the angle BAC, whose equation is $my + nz = 0$. The tangent to the cubic at the point D is $l(lx + my + nz) = k^2x$, which is the anti-parallel of DA with respect to the angle BDC. The three conics
$$k^2/(lx + my + nz) + m/y + n/z = 0, \quad k^2/(lx + my + nz) + n/y + l/x = 0,$$
$$k^2/(lx + my + nz) + l/x + m/y = 0$$
have triple contact with the cubic at the points A, E, F; B, F, D; C, D, E respectively.

Digitized by Google

It is seen at once that, if $k^2 = 0$, the locus reduces to the line $lx + my + ns = 0$ and the conic $l/x + m/y + n/s = 0$.

It can be shown that, if $k^2 = 0$, we must have $l/a = m/b = n/c$; so that the degenerate cubic is composed of the line at infinity and the circle ABC. In this case the generating conic touches the line at infinity, and therefore is a parabola.

It may be interesting to observe that there are other cases in which the cubic degenerates. It is clear that, if the line AD be at once the bisector of the angles BAC and BDC, then AD is part of the cubic. The conditions that this may be the case are $m + n = 0$ and $k = l$, when the cubic reduces to $(y - z)(lyz - lx^2 + myx - mxy) = 0$ where $ma = l(b - c)$.

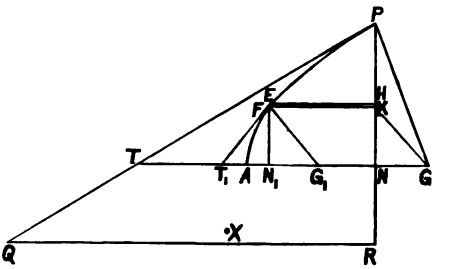
If the four lines form a parallelogram, the cubic reduces to the line at infinity and a conic circumscribing the parallelogram. If the parallelogram be a rhombus, this conic reduces to the two diagonals of the parallelogram.

Note.—Professor Sanjána and Mr. A. R. Richardson refer to Salmon's *Conics* for other methods of dealing with the problem.

11011. (Professor Crofton, F.R.S.)—Prove that the mean distances of the vertex of a triangle from all points in the area is equal to its distance from the centroid, *measured along a parabolic path*, which leaves the vertex in the direction of one of the sides, and reaches the centroid in a direction parallel to the others, the axis of the parabola being parallel to the base.

Note by R. Chartres.

By taking the vertex A of the parabola to be the centroid of the triangle PQR, we get a very elementary proof of this famous problem.



Since the sub-normal is constant, EF = HK.KG/NG by similar triangles; therefore

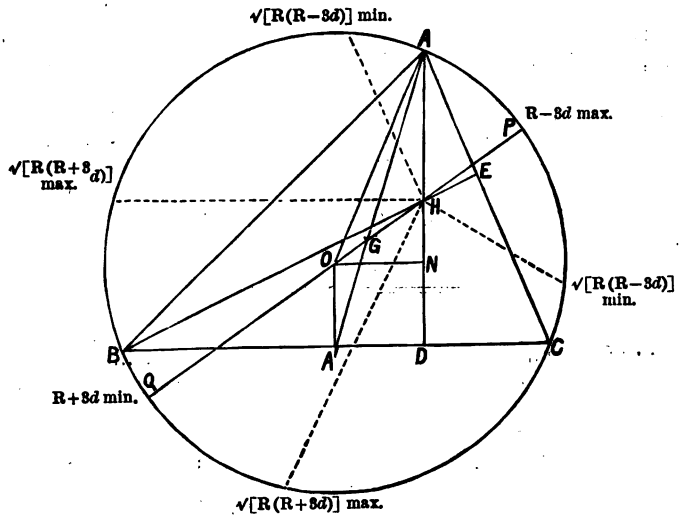
arc AP = PN/NG (mean distance of all points in PN from G)
= mean distance of all points in TN from P by similar triangles
= M(PX).

Digitized by Google

15542. (D. Biddle.)—A triangle is inscribed in a circle of given radius, and has its centroid at a given distance from the centre. (1) Define the limits of area of inscribed triangles whose centroids are thus placed; and (2), given any area within those limits, inscribe a second triangle having this area, and whose centroid occupies the same position as that of the former or one equidistant from the centre of the circle.

Solution by Professor Sanjána, M.A.

Let R be the given radius, and d the given distance from the centre (O) of the centroid (G). Join OG, produce it, and make GH = 2OG;



then H is the orthocentre of the triangle. Taking any point of the circumference A as vertex, the inscribed triangle may be thus obtained:—Join AH, draw OA′ parallel to and half of AH, through A′ draw BC at right angles to OA′ and AH produced; then ABC is the required triangle. Thus, H being fixed, the variable area may be taken as a function of AH. Let AH = $2x$, so that OA′ = x; then BC = $2\sqrt{(R^2-x^2)}$. Also HN = $(R^2-4x^2-9d^2)/4x$; hence

$$AD = 2x + HN + x = (8x^2 + R^2 - 9d^2)/4x.$$

Therefore four times the area of the triangle

$$= 2 \cdot BC \cdot AD = (8x^2 + R^2 - 9d^2)\sqrt{(R^2-x^2)}/x = f(x) \text{ suppose.}$$

It is found that $f'(x) = (9d^2R^2 - R^4 + 8x^2R^2 - 16x^4)/\{x^2\sqrt{(R^2-x^2)}\}$.

Equating the numerator to zero, we get $4x^2 = R^2 \pm 3dR$. Taking positive values only, we have, if $2x < \sqrt{(R^2-3dR)}$, $f'(x)$ negative; if $2x > \sqrt{(R^2-3dR)}$ and $< \sqrt{(R^2+3dR)}$, $f'(x)$ positive; if $2x > \sqrt{(R^2+3dR)}$, $f'(x)$ negative. Hence $4x^2 = R^2 - 3dR$ determines a minimum value,

Digitized by Google

viz., $\frac{4}{9}\sqrt{3}(R-3d)^{\frac{1}{2}}(R+d)^{\frac{3}{2}}$, for the area; and $4x^2 = R^2 + 3dR$ determines a maximum value, viz., $\frac{4}{9}\sqrt{3}(R+3d)^{\frac{1}{2}}(R-d)^{\frac{3}{2}}$.

Again, when $2x = R-3d$, $\frac{1}{4}f(x) = \frac{4}{9}\sqrt{3}(R+3d)^{\frac{1}{2}}(R-d)^{\frac{3}{2}}$, the above maximum; and, when $2x = R+3d$, $\frac{1}{4}f(x) =$ the above minimum. The maxima and minima values are obtained therefore for six alternate positions of A as marked in the figure. The reason why $2x = R \pm 3d$ is not obtained from $f'(x) = 0$ is that x is not a perfectly independent variable, but has itself maxima and minima values at P and Q, the extremities of the diameter through O and G; see Edwards, *Diff. Cal.*, § 492.

If any given value of the area Δ, between the limits

$$\frac{4}{9}\sqrt{3}(R-3d)^{\frac{1}{2}}(R+d)^{\frac{3}{2}} \quad \text{and} \quad \frac{4}{9}\sqrt{3}(R+3d)^{\frac{1}{2}}(R-d)^{\frac{3}{2}},$$

be taken, to find the triangle we have to solve $f(x) = 4\Delta$, and obtain $4x^2$, *i.e.*, $2x$; with this as radius, describe about H as centre a circle cutting the given circumference in two points, either of which will serve for A′, the apex of the second triangle; the triangle A′B′C′ is then easily constructed, with A′ as vertex, H as orthocentre, and O as circum-centre. The diameter OG may be fixed in direction without altering the generality of the problem; the same triangle will always be obtained, though the position may change by taking G on the circumference of a circle about O as centre.

14735. (Professor Morley.)—Given five planes $a_1, \ldots, a_5$, and a quadric Q, let a quadric Q_n pass through the intersection of Q and a_n ($n = 1, \ldots, 5$) and the four intersections of the four other planes. Then Q_n intersects Q in a second plane β_n, whose polar as to Q is a point α_n. Prove that the line a_1a_2 passes through the point (a_3, a_4, a_5); or, reciprocally, that the line a_1a_2 lies in the plane $a_3a_4a_5$. Also prove that, if one of the points a lies on Q, then all the points a lie on Q. Arrange the proof so as to hold for the analogous theorem in n dimensions. [The theorem in two dimensions that, given four lines and a conic C, and conics C_n being passed through the three intersections of three lines and the intersections of C and the fourth line, then, if one C_n touches C, all C_n's touch C, is M. Humbert's, *Comptes Rendus*, Vol. CXXIX.; and an elegant elementary proof is given by M. Bricard, *Nouv. Ann.*, Sér. 3, Vol. XIX., p. 369.]

Additional Solution by Professor Nanson.

The planes α are connected by an identical relation, which may be taken to be $\Sigma a_r = 0$, and in virtue of it Q may be expressed in the form $\Sigma Q_{rs}a_ra_s$, where Q_{rs} ($= Q_{sr}$) is a constant, and $Q_{rr} = 0$. Now, any quadric through (a_1, Q) is $Q = a_1\Sigma\lambda_{1r}a_r$, where $\lambda_{11} = 0$; and, expressing that it goes through the meets of $a_2, a_3, \ldots, a_n$, we get $\lambda_{1r} = Q_{1r}$; so that $\beta_1 \equiv \Sigma Q_{1r}a_r$. Now, we have to show that the polar line a_1a_2 of $\beta_1\beta_2$ goes through the point $(a_3, a_4, \ldots, a_n)$, or, what is the same thing, that the polar plane of $a_3a_4 \ldots a_n$ goes through $\beta_1\beta_2$, and this is obvious, because this plane is $a_1'\Sigma Q_{2r}a_r + a_2'\Sigma Q_{1r}a_r = 0$ or $a_1'\beta_2 + a_2'\beta_1 = 0$, where a_1', a_2' are the values of a_1, a_2 at the point $(a_3, a_4, \ldots, a_n)$.

Digitized by Google

If the point a_n is on Q, then the plane β_n touches Q. Hence, since $Q \equiv a_n\beta_n + \Sigma Q_{pq} a_p a_q$, where $p, q = 1, 2, \ldots, n-1$, it follows that β_n touches $\Sigma Q_{pq} a_p a_q = 0$, and the bordered determinant condition for this is $|Q_{rs}| = 0$, the symmetry of which shows that all the points a are on Q.

15575. (R. Chartres.)—If $mx - 1 = r \cdot s$, $x > 3$, then, if (1) r and s are consecutive integers, r the greater, $1/x$ will give six recurring (*bona fide*) figures in the scales $\left(nx\begin{smallmatrix}+r\\-s\end{smallmatrix}\right)$, and three recurring (*bona fide*) figures in the scales $\left(nx\begin{smallmatrix}+s\\-r\end{smallmatrix}\right)$; (2) but, if $r = s$, $1/x$ will give four such figures in the scales $(nx \pm r)$. Proof required without the modern theory of numbers.

Solution by the Proposer.

$$r^6 - 1 = (r-1)(r+1)(r^2+r+1)(r^2-r+1).$$

Omitting the first three factors, which give only 1, 2, 3 recurring figures respectively, then $1/x = 1/(r^2-r+1) = 1/(rs+1)$, where $r = s+1$ gives six recurring figures in the scales $\left(nx\begin{smallmatrix}+r\\-s\end{smallmatrix}\right)$ (Question 15462), and, if $mx-1 = rs$, $1/x$ will therefore give six recurring figures. Similarly we get three recurring figures in the scales $\left(nx\begin{smallmatrix}+s\\-r\end{smallmatrix}\right)$, and four if $r = s$ in the scales $(nx \pm r)$.

(1) Since $3 \cdot 19 - 1 = 8 \cdot 7$, therefore $\frac{1}{19}$ will give six figures in the scales $\left(19n\begin{smallmatrix}+8\\-7\end{smallmatrix}\right)$, and three figures in the scales $\left(19n\begin{smallmatrix}+7\\-8\end{smallmatrix}\right)$.

(2) If $x = r^2+1$, then $1/x = \cdot\dot{0}0\,(r-1)(r-\dot{1})$. Hence, if we might use an irrational radix, $\frac{1}{19} = 1/[(\sqrt{18})^2+1] = \cdot\dot{0}0\,(3\sqrt{2}-1)(3\sqrt{2}-\dot{1})$, or four figures in the scale $(3\sqrt{2})$.

15554. (J. J. Barniville, B.A.)—Having $u_n + u_{n+3} = 3u_{n+1} + 3u_{n+2}$, prove that

$$\left(\frac{5}{-1.7}\right)^2 - \left(\frac{13}{7.17}\right)^2 + \left(\frac{53}{17.73}\right)^2 - \left(\frac{193}{73.263}\right)^2 + \ldots = \frac{1}{2},$$

$$\left(\frac{5}{1.7}\right)^2 - \left(\frac{17}{7.23}\right)^2 + \frac{65}{23.89}\Big)^2 - \left(\frac{241}{89.329}\right)^2 + \ldots = \frac{1}{2},$$

$$\left(\frac{13}{-7.17}\right)^2 - \left(\frac{25}{17.31}\right)^2 + \left(\frac{109}{31.51}\right)^2 - \left(\frac{389}{151.529}\right)^2 + \ldots = \frac{1}{98},$$

$$\left(\frac{17}{-7.23}\right)^2 - \left(\frac{37}{23.47}\right)^2 + \left(\frac{151}{47.217}\right)^2 - \left(\frac{565}{217.769}\right)^2 + \ldots = \frac{1}{98}.$$

Digitized by Google

Solution by Professor S. Sircom.

We have

$$\left(\frac{5}{-1.7}\right)^2-\left(\frac{13}{7.17}\right)^2=\left(\frac{5}{-1.7}-\frac{13}{7.17}\right)\left(\frac{5}{-1.7}+\frac{13}{7.17}\right)=\frac{98.72}{1^2.7^2.17^2}$$
$$=\frac{144}{1^2.17^2}=\tfrac{1}{2}\frac{288}{1^2.17^2}=\tfrac{1}{2}\left(1-\frac{1}{17^2}\right).$$

Similarly $\left(\frac{53}{17.73}\right)^2-\left(\frac{193}{73.263}\right)^2=\frac{1}{2}\left(\frac{1}{17^2}-\frac{1}{263^2}\right)$,

and the sum of the first series is $\frac{1}{2}$. The other results are obtained in the same manner: thus

$$\left(\frac{13}{7.17}-\frac{25}{17.31}\right)\left(\frac{13}{7.17}+\frac{25}{17.31}\right)=\frac{228.578}{7^2.17^2.31^2}=\frac{456}{7^2.31^2}$$
$$=\tfrac{1}{2}\left(\frac{1}{7^2}-\frac{1}{31^2}\right),$$

and the sum of the third series is $\frac{1}{2}\frac{1}{7}$.

To prove the property upon which the summation depends we have to show that

$u_{n-2}v_n-u_{n-1}v_{n-2}=2v_{n-1}^2$, $u_{n-2}v_n+u_{n-1}v_{n-2}=\frac{1}{2}(v_n^2-v_{n-1}^2)$ (n being even),

where $u_0, u_1, u_2, \ldots$ is the series 5, 13, 53 .....(1),

and $v_0, v_1, v_2, \ldots$ is the series -1, 7, 17 .....(2).

The generating function of (1) is

$$\frac{5-2x-x^2}{(1+x)(1-4x+x^2)}=\frac{1}{1+x}+2\frac{2-x}{1-4x+x^2},$$

$$\frac{2-x}{1-4x+x^2}=\frac{2-a^{-1}}{1-a^{-2}}\frac{1}{1-ax}+\frac{a-2}{a^2-1}\frac{1}{1-a^{-1}x}$$

(where $a=2+\sqrt{3}$, $a^{-1}=2-\sqrt{3}$)

$$=\frac{2a-1}{a-a^{-1}}\frac{1}{1+ax}+\frac{1-2a^{-1}}{a-a^{-1}}\frac{1}{1-a^{-1}x}=\tfrac{1}{2}\frac{a}{1-ax}+\tfrac{1}{2}\frac{a^{-1}}{1-a^{-1}x},$$

whence $u_{n-2}=a^{n-1}+1+a^{-n+1}$; $u_{n-1}=a^n-1+a^{-n}$. The generating function of (2) is $\frac{10x-x^2-1}{(1+x)(1-4x+x^2)}=\frac{1+x}{1-4x+x^2}-\frac{2}{1+x}$,

and, proceeding as before,

$$v_n=\tfrac{1}{2}(\sqrt{3}+1)a^n-2+\tfrac{1}{2}(\sqrt{3}-1)a^{-n},$$
$$v_{n-1}=\tfrac{1}{2}(\sqrt{3}+1)a^{n-1}+2+\tfrac{1}{2}(\sqrt{3}-1)a^{-n+1},$$
$$v_{n-2}=\tfrac{1}{2}(\sqrt{3}+1)a^{n-2}-2+\tfrac{1}{2}(\sqrt{3}-1)a^{-n+2},$$

which gives the required results on substitution.

15276. (θ. β.)—Let x, y be circular co-ordinates, and let the equations to three lines be $xt_s+y=x_st_s$ ($s=1, 2, 3$), where $|t_s|=1$ and $x=x_s$, $y=x_st_s$ is the reflection of the origin in the line. Verify that, for the triangle formed of these lines, the centroid is given by $x=a_1-\frac{1}{2}a_2p$, the

Digitized by Google

orthocentre by $x = a_1 - a_2 p$, and the Simson-line of the point $x = a_1 - a_2 t$ by $$xt^2 - yt = -\tfrac{1}{2}a_2 t^3 + (a_1 - \tfrac{1}{2}a_2 p)t^2 - (b_1 - \tfrac{1}{2}a_2 q)\,t + a_2 r,$$ where $$a_1 = \Sigma x_1 t_1^3/(t_1-t_2)(t_1-t_3), \quad a_2 = \Sigma x_1 t_1/(t_1-t_2)(t_1-t_3),$$ $$b_1 = r\Sigma x_1/(t_1-t_2)(t_1-t_3),$$ $$p = t_1+t_2+t_3, \quad q = t_2t_3+t_3t_1+t_1t_2, \quad r = t_1t_2t_3.$$

[For notation and methods, see Morley, *Transactions of the American Mathematical Society*, Vol. I., p. 97.] From this expression for the Simson-line we obtain immediately the well known result that its envelope is a hypocycloid of class 3, with the nine-point circle as its vertex-circle.

Solution by G. N. Bates, M.A.

The intersection of the lines (ii.) and (iii.) is given by

$$x = \frac{x_2t_2 - x_3t_3}{t_2 - t_3} = a_1 - a_2 t_1, \quad y = -\frac{t_2t_3(x_2 - x_3)}{t_2 - t_3} = b_1 - \frac{ra_2}{t_1};$$

therefore the circum-centre of the triangle formed by the three lines is $x = a_1$, while $|a_2|$ is the circum-radius. The centroid is therefore $x = a_1 - \frac{1}{3}a_2 p$, and the orthocentre $x = a_1 - a_2 p$.

The angle θ between the two lines

$$x\lambda_1 + y = \text{constant} \quad \text{and} \quad x\lambda_2 + y = \text{constant}$$

is given by $e^{2i\theta} = \lambda_1/\lambda_2$. Now the angle between the Simson-line of P and BC is the complement of half the angle subtended at O by the arc PA of the circum-circle, and the Simson-line also passes through the mid-point of PH. Its equation is therefore

$$xt - y = a_1 t - \tfrac{1}{2}a_2 t\,(t+p) - b_1 + ra_2/2t + \tfrac{1}{2}a_2 q \quad \text{.....(i.)}$$

or $$xt^2 - ty = -\tfrac{1}{2}a_2 t^3 + (a_1 - \tfrac{1}{2}a_2 p)\,t^2 - (b_1 - \tfrac{1}{2}a_2 q)\,t + \tfrac{1}{2}ra_2 \quad \text{.....(ii.).}$$

We may now either obtain the (xy) equation of the envelope by differentiating (ii.) with respect to t, or the (xt) equation of the envelope by differentiating (i.) with respect to t. We have then, on differentiating (i.),

$$x = a_1 - \tfrac{1}{2}a_2 p - a_2 t - ra_2/2t^2 \quad \text{or} \quad x - a_1 + \tfrac{1}{2}a_2 p = -a_2 t - ra_2/2t^2,$$

which is found to agree with the equation of a three-cusped hypocycloid whose centre is at the point $a_1 - \frac{1}{2}a_2 p$ (the N.P. centre.

[*Cf.*, for the more general theorem when lines inclined at θ to the sides are taken, a paper "On a System of Hypocycloids of Class 3 inscribed in a given 3-line, &c.," by Dr. H. A. Converse, *Annals of Mathematics*, April, 1904.]

N.B.—My equation (ii.) differs from the equation of the Simson-line in Question in having $\frac{1}{2}ra_2$ for ra_2.

15563. (R. F. Davis, M.A.)—Given a hyperbola whose eccentricity = 2, prove that the circle described upon any focal chord as diameter passes through the further extremity of the transverse axis.

Digitized by Google

Solutions (I.) *by* R. TUCKER, M.A., (II.) *by* C. M. ROSS.

(I.) Referring to A′ as origin, the equation to the curve is

$$x^2/a^2 - 2x/a = y^2/b^2 \quad \text{.....(i.).}$$

Let the equation to PQ be

$$mx = a(m-y)(1+e) \quad \text{.....(ii.);}$$

then the equation to the two straight lines A′P, A′Q will be

$$a^2m(1+e)(x^2/a^2 - y^2/b^2) = 2x[mx + ay(1+e)],$$

whence $$b^2m[(1+e)-2]x^2 \ldots - a^2m(1+e)y^2 = 0.$$

These lines are at right angles if $b^2(e-1) - a^2(1+e) = 0$, *i.e.*, if $e^2 - e - 2 = 0 = (e-2)(e+1)$. Hence, &c.

(II.) The equation of the hyperbola is

$$3x^2 - y^2 = 3a^2 \quad \text{.....(1),}$$

and the equation of a focal chord is

$$y = m(x-2a) \quad \text{.....(2).}$$

(2) cuts (1) in the points whose abscissæ are given by

$$(m^2-3)x^2 - 4am^2x + a^2(4m^2+3) = 0.$$

If x_1 and x_2 are the roots of this equation,

$$x_1 + x_2 = 4am^2/(m^2-3), \quad x_1x_2 = a^2(4m^2+3)/(m^2-3).$$

Again, if y_1 and y_2 are the ordinates of ends of the focal chord,

$$y_1 + y_2 = 12am/(m^2-3) \quad \text{and} \quad y_1y_2 = 9a^2m^2/(m^2-3).$$

Hence the equation of the circle on focal chord as diameter is

$$(m^2-3)(x^2+y^2) - 4am(mx+3y) - a^2(5m^2-3) = 0.$$

This circle plainly passes through the point $(-a, 0)$, which is the further extremity of the transverse axis.

15539. (Lt.-Col. ALLAN CUNNINGHAM, R.E.)—For what forms of m is $X^2 \pm 1 \equiv 0 \pmod{m}$? Show how to find X when m is given. Ex.—$m = 99991$.

Solution by the PROPOSER.

(i.) Excluding the obvious divisor $m = X \pm 1$, all *prime* divisors must be of form $p = 6w + 1$, and all *composite* divisors must be a product of such primes, so that $m = A^2 + 3B^2$.

(ii.) When m is *given*, let m be expressed in the form $m = A^2 + 3B^2$.

Then $$m = \frac{(A-B)^2 - (2B)^2}{(A-B) - (2B)} = \frac{x^2 - y^2}{x - y} \quad \text{(suppose).}$$

Here $x = A - B$, $y = 2B$;

$$x^2 - y^2 \equiv 0 \pmod{m};$$

therefore $$(x \pm km)^2 \equiv y^2 \pmod{m}.$$

Now find k such that $x \pm km \equiv 0 \pmod{m}$, so that

$$(x \pm km)/y = \text{integer} = X.$$

Digitized by Google

Then X (when positive) is a solution of $X^3-1 \equiv 0$, and X (when negative) is a solution of $X^3+1 \equiv 0 \pmod m$.

Ex.—$m = 99991 = 278^2+3.87^2 = A^2+3B^2$. Therefore

$x = A-B = 191$, $y = 2B = 174 = 2.3.29$, and $191^3 \equiv 174^3 \pmod m$.

Taking $k = 17$, $191+17m = 1700038 = 2.29.29311$; therefore

$$(2.29.29311)^3 \equiv (2.3.29)^3, \quad \text{and} \quad 29311^3 \equiv 3^3 \pmod m;$$

therefore

$$(29311-m)^3 \equiv -70680^3 \equiv 3^3, \quad \text{and} \quad 23560^3 \equiv -1 \pmod m.$$

Thus m is a divisor of (23560^3+1); and therefore also of (23559^3-1).

Note.—The writer's *Tables of Quadratic Partitions* (now in the press) give the partitions $p = A^2+3B^2$ for all primes $p \not> 100{,}000$; so that the roots of $x^3 \pm 1 \equiv 0 \pmod p$ can be found by these tables up to

$$p \not> 100{,}000.$$

15375. (Professor Nanson.)—Prove that

$$(a_1X_2Y_3Z_4)(f_1X_2Y_3Z_4)+b_1(X_2Y_3Z_4)(g_1X_2Y_3Z_4)+(c_1X_2Y_3Z_4)(h_1X_2Y_3Z_4) = 0$$

where $X_p,\ Y_p,\ Z_p,\ = h_py-g_pz+a_pw$; $f_pz-h_px+b_pw$; $g_px-f_py+c_pw$.

Solution by the Proposer.

The determinant aggregate is, with sign changed, manifestly half the seven-line determinant

$$\begin{vmatrix} (\lambda_{pq}) & X_p & Y_p & Z_p \\ X_q & . & . & . \\ Y_q & . & . & . \\ Z_q & . & . & . \end{vmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

where (λ_{pq}) is the compound of the four-row arrays

$$(a_r b_r c_r f_r g_r h_r), \ (f_r g_r h_r a_r b_r c_r) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

and therefore vanishes because (1) is the compound of the deficient arrays obtained by placing under the arrays (2) respectively the arrays

$$\begin{matrix} . & -z & y & w & . & . \\ z & . & -x & . & w & . \\ -y & x & . & . & . & w \end{matrix} \quad ; \quad \begin{matrix} w & . & . & . & -z & y \\ . & w & . & z & . & -x \\ . & . & w & -y & x & . \end{matrix} \quad \ldots (3).$$

It is obvious that this result may be generalized by replacing the arrays (2) by any two arrays A, A′, and the arrays (3) by any two arrays B, B′ whose compound is an array of zeros, provided that all the arrays A, A′, B, B′ have the same number of columns and that there are as many rows in A′ and B′ as in A and B. It is manifest that this generalisation includes as a very special case the result of Question 15426; *cf.* Vol. v., New Series, p. 75.

Digitized by Google

Calcul des côtés (1) *du triangle et hexagone réguliers;* (2) *des décagones.*

By Professor L. Catalá.

(1) Soient des arcs $AB = BC = CD = \frac{1}{6}\pi$; alors $AB = \lambda_6$ et $BD = \lambda_3$. Le triangle rectangle ABD donne

$$\lambda_6^2 + \lambda_3^2 = 4 \quad \text{.....(i.),}$$

et le quadrilatère cyclique ABCD,

$$\lambda_6^2 + 2\lambda_6 = \lambda_3^2 \quad \text{.....(ii.).}$$

Éliminant λ_3 entre (i.) et (ii.), $\lambda_6^2 + \lambda_6 - 2 = 0$. Des deux racines $+1$ et -2 de cette équation, la racine positive seule convient, et l'on a $\lambda_6 = 1$. L'élimination de λ_6 entre (i.) et (ii.) conduit à l'équation $4\lambda_3(\lambda_3^2 - 3) = 0$. L'unique racine admissible est $\lambda_3 = \sqrt{3}$. On trouve ainsi la valeur du côté du triangle équilatéral, sans supposer connu celle du hexagone.

(2) Soient les arcs $AB = BC = CD = DE = EF = \frac{1}{10}\pi$. Dans le triangle rectangle ADF, $\lambda_{5,1}^2 + \lambda_{10,3}^2 = 4$ .....(i.), et dans le quadrilatère cyclique ACDF,

$$\lambda_{10,3}^2 = \lambda_{5,1}^2 + 2\lambda_{10,1} \text{.....(ii.).}$$

De (i.) et (ii.),

$$\lambda_{10,3}^2 = 2 + \lambda_{10,1} \text{.....(iii.).}$$

D'ailleurs, M étant le point d'intersection de FC et DE, des triangles isoscèles MFE et COM, $\lambda_{10,3} - \lambda_{10,1} = 1$. De cette équation et (iii.),

$$\lambda_{10,3}^2 - \lambda_{10,3} - 1 = 0.$$

Tirant d'ici $\lambda_{10,3}$, puis substituant dans (iii.),

$$\lambda_{10,3} = \tfrac{1}{2}(\sqrt{5}+1), \qquad \lambda_{10,1} = \tfrac{1}{2}(\sqrt{5}-1),$$

en ne considérant que la racine positive de la dernière équation.

15517. (C. M. Ross.)—Eliminate x, y, z from the equations

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0,$$
$$a'x^2 + b'y^2 + c'z^2 + 2f'yz + 2g'zx + 2h'xy = 0,$$
$$a''x^2 + b''y^2 + c''z^2 + 2f''yz + 2g''zx + 2h''xy = 0,$$

expressing the eliminant as the equation of a determinant to zero.

Solution by the Proposer.

Let
$$u \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0 \text{.....(1),}$$
$$v \equiv a'x^2 + b'y^2 + c'z^2 + 2f'yz + 2g'zx + 2h'xy = 0 \text{.....(2),}$$
$$w \equiv a''x^2 + b''y^2 + c''z^2 + 2f''yz + 2g''zx + 2h''xy = 0 \text{.....(3).}$$

Then $\partial u/\partial x = 2(ax + hy + gz) = 0$, $\partial u/\partial y = 2(hx + by + fz) = 0$,
$\partial u/\partial z = 2(gx + fy + cz) = 0$, ...

Hence

$$J \equiv \begin{vmatrix} ax+hy+gz, & hx+by+fz, & gx+fy+cz \\ a'x+h'y+g'z, & h'x+b'y+f'z, & g'x+f'y+c'z \\ a''x+h''y+g''z, & h''x+b''y+f''z, & g''x+f''y+c''z \end{vmatrix} = 0.$$

Digitized by Google

The Jacobian is evidently of the third degree; it is therefore of the form

$$J \equiv Ax^3 + By^3 + Cz^3 + 3Fx^2y + 3Gxy^2 + 3Hy^2z + 3Kyz^2 + 3Lz^2x + 3Mzx^2 + 6Nxyz = 0.$$

Again

$$\partial J/\partial x = Ax^2 + Gy^2 + Lz^2 + 2Nyz + 2Mzx + 2Fxy = 0 \quad \ldots\ldots(4),$$

$$\partial J/\partial y = Fx^2 + By^2 + Kz^2 + 2Hyz + 2Nzx + 2Gxy = 0 \quad \ldots\ldots(5),$$

$$\partial J/\partial z = Mx^2 + Hy^2 + Cz^2 + 2Kyz + 2Lzx + 2Nxy = 0 \quad \ldots\ldots(6).$$

Hence, eliminating x^2, y^2, z^2, $2yz$, $2zx$, $2xy$ from equations (1), ..., (6), the required eliminant is

$$\begin{vmatrix} a & b & c & f & g & h \\ a' & b' & c' & f' & g' & h' \\ a'' & b'' & c'' & f'' & g'' & h'' \\ A & G & L & N & M & F \\ F & B & K & H & N & G \\ M & H & C & K & L & N \end{vmatrix} = 0.$$

9700. (Professor Hanumanta Rau.)—Prove that, if n be even,

$$\sec^2 a + \sec^2(a + \pi/n) + \sec^2(a + 2\pi/n) + \ldots \text{ to } 2n \text{ terms} = 2n^2 \operatorname{cosec}^2 na.$$

Solution by Professor G. B. M. Zerr.

From trigonometry we get

$$\sin a \sin(a + \pi/n) \sin(a + 2\pi/n) \ldots \sin\{a + (n-1)\pi/n\} = \sin na/2^{n-1} \quad (1).$$

Put $a = \pi + a$ in (1); then, if n be even,

$$\sin(\pi + a) \sin\{a + (n+1)\pi/n\} \sin\{a + (n+2)\pi/n\} \ldots \sin\{a + (2n-1)\pi/n\} = \sin na/2^{n-1} \ldots(2).$$

The product of (1) and (2) gives

$$\sin a \sin(a + \pi/n) \sin(a + 2\pi/n) \ldots \sin\{a + (2n-1)\pi/n\} = \sin^2 na/2^{2n-2} \ldots(3).$$

Put $a = a + \frac{1}{2}\pi$ in (3), and we get, if n be even,

$$\cos a \cos(a + \pi/n) \cos(a + 2\pi/n) \ldots \cos\{a + (2n-1)\pi/n\} = \sin^2 na \cos^2 \tfrac{1}{2}n\pi/2^{2n-2} = \sin^2 na/2^{2n-2} \ldots(4).$$

Differentiate (4) with respect to a, and we get

$$\sin^2 na/2^{2n-2}\{\tan a + \tan(a + \pi/n) + \tan(a + 2\pi/n) + \ldots \text{ to } 2n \text{ terms}\} = -(2n \sin nd \cos na)/2^{2n-2};$$

therefore

$$\tan a + \tan(a + \pi/n) + \tan(a + 2\pi/n) + \ldots \text{ to } 2n \text{ terms} = -2n \cot na \ldots(5).$$

Differentiating (5) with respect to a,

$$\sec^2 a + \sec^2(a + \pi/n) + \sec^2(a + 2\pi/n) + \ldots \text{ to } 2n \text{ terms} = 2n^2 \operatorname{cosec}^2 na.$$

Note on the Solution of certain Equations.

By A. M. Nesbitt, M.A.

I do not know whether the following point has hitherto escaped attention; at any rate I have never seen it noticed.

Digitized by Google

In solving the equations

$$\frac{x}{a+\lambda}+\frac{y}{b+\lambda}+\frac{z}{c+\lambda}=1, \qquad \frac{x}{a+\mu}+\frac{y}{b+\mu}+\frac{z}{c+\mu}=1,$$

$$\frac{x}{a+\nu}+\frac{y}{b+\nu}+\frac{z}{c+\nu}=1,$$

the usual method adopted is to show that the "cubic" in θ,

$$\frac{x}{a+\theta}+\frac{y}{b+\theta}+\frac{z}{c+\theta}-1=\frac{(\lambda-\theta)(\mu-\theta)(\nu-\theta)}{(a+\theta)(b+\theta)(c+\theta)},$$

is satisfied by the *four* values $\lambda, \mu, \nu, \infty$ for θ, and is therefore an identity.

Now, in actual fact, the terms independent of θ vanish at once; for there are $bcx+cay+abz-abc$ on the left-hand side, and $\lambda\mu\nu$ on the right-hand side; while, since λ, μ, ν are the three roots of any one of the three proposed equations, we get at once $\lambda\mu\nu=\Sigma bcx-abc$. Thus, the "cubic" has a root 0, and the resulting "quadratic" still has the three roots λ, μ, ν. The "infinite root" has never seemed entirely satisfactory to me.

8735. (A. Gordon.)—If

$$(x^2/a^2+y^2/b^2+z^2/c^2-1)\,\theta+(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2=0$$

represent a cone, prove that $a^2+\theta$ is one of the primary axes of the confocals through (α, β, γ).

Solution by Frances E. Cave.

The condition for a cone is $\Sigma\dfrac{a^2}{1+\theta/a^2}-\Sigma a^2+\theta=0$; therefore

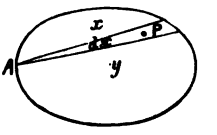
$$-\Sigma\frac{a^2\theta}{a^2+\theta}+\theta=0.$$

Therefore (α, β, γ) lies on $\dfrac{x^2}{a^2+\theta}+\dfrac{y^2}{b^2+\theta}+\dfrac{z^2}{c^2+\theta}=1$.

10173. (Rev. T. C. Simmons, M.A.)—Within a given closed convex area a random point is taken, and joined with any point on the perimeter. Show that, if two more random points be now taken in the area, the odds are two to one in favour of their falling on the same side of the line thus drawn. [This is Question 8604.]

Solution by R. Chartres.

Let Δ denote the area. For all positions of P in the element dx the value of x is the same; therefore $\mathrm{M}(x^2)=\frac{1}{3}\Delta^2$ and $\mathrm{M}(xy)=\frac{1}{6}\Delta^2$, or the chance of both the other random points being on the same side of AP is twice that of being one on each side.



Digitized by Google

6487. (Rev. Preb. W. A. Whitworth, M.A.).—Express all the numbers from 1 to 140 by four nines. Algebraical symbols and decimal points may be used, but the expression for each number must contain the four nines *and no other figure*. Also express the same series of numbers by four fours. [As an example of what is required, the Proposer gives the number 34 thus:—$34 = (99 + \sqrt{9})/\sqrt{9} = 4.4.\sqrt{4} + \sqrt{4}$.]

[N.B.—Only the tables of solution by four *nines* are in each case published here. The solution of a similar problem by means of three *fours* is already in print, *vide* note on Solution (I.).—Editor.]

Solutions (I.) *by* Lt.-Col. Allan Cunningham, R.E.;
(II.) *by* Rev. T. Wiggins, B.A.

	(I.)	(II.)
n	*Four nines.*	*Four nines.*
0	$9-9+9-9$	
1	$99/99$	$\cdot\dot{9}9\dot{9}\dot{9}$
2	$9/9+9/9$	$\cdot\dot{9}\dot{9}+\cdot\dot{9}\dot{9}$
3	$(9+9+9)/9$	$\cdot\dot{9}+\cdot\dot{9}+\cdot\dot{9}\dot{9}$
4	$\cdot\dot{9}+\cdot\dot{9}+\cdot\dot{9}+\cdot\dot{9}$	$\cdot\dot{9}+\cdot\dot{9}+\cdot\dot{9}+\cdot\dot{9}$
5	$9-\sqrt{9}-9/9$	$\sqrt{9}+\cdot\dot{9}+\cdot\dot{9}\dot{9}$
6	$9+9-9-\sqrt{9}$	$\sqrt{9}+\cdot\dot{9}+\cdot\dot{9}+\cdot\dot{9}$
7	$9-\sqrt{9}+9/9$	$\sqrt{9}+\sqrt{9}+\cdot\dot{9}\dot{9}$
8	$9+9-9-\cdot\dot{9}$	$\sqrt{9}+\sqrt{9}+\cdot\dot{9}+\cdot\dot{9}$
9	$9+9(9-9)$	$9.\cdot\dot{9}\dot{9}\dot{9}$
10	$(99-9)/9$	$9\cdot\dot{9}\dot{9}\dot{9}$
11	$9+\sqrt{9}-9/9$	$9\cdot\dot{9}+\cdot\dot{9}\dot{9}$
12	$(99+9)/9$	$9\cdot\dot{9}+\cdot\dot{9}+\cdot\dot{9}$
13	$9+\sqrt{9}+9/9$	$9\cdot\dot{9}\dot{9}+\sqrt{9}$
14	$99/9+\sqrt{9}$	$9\cdot\dot{9}+\cdot\dot{9}+\sqrt{9}$
15	$9+9-9/\sqrt{9}$	$9.\cdot\dot{9}+\sqrt{9}+\sqrt{9}$
16	$9+\sqrt{9}+\sqrt{9}+\cdot\dot{9}$	$9+\cdot\dot{9}+\sqrt{9}+\sqrt{9}$
17	$9+9-9/9$	$9+9-\cdot\dot{9}\dot{9}$
18	$9+9+9-9$	$9+9+9-9$
19	$9+9+9/9$	$9+9+\cdot\dot{9}\dot{9}$
20	$99/9+9$	$9+9+\sqrt{9}-\cdot\dot{9}$
21	$9+9+9/\sqrt{9}$	$9.\cdot\dot{9}+9+\sqrt{9}$
22	$9+9+\sqrt{9}+\cdot\dot{9}$	$9+9+\sqrt{9}+\cdot\dot{9}$
23	$9\sqrt{9}-\sqrt{9}-\cdot\dot{9}$	$9\sqrt{9}-\sqrt{9}-\cdot\dot{9}$
24	$9+9+9-\sqrt{9}$	$9+9+9-\sqrt{9}$
25	$9\sqrt{9}-\cdot\dot{9}-\cdot\dot{9}$	$9\sqrt{9}-\cdot\dot{9}-\cdot\dot{9}$
26	$9\sqrt{9}-9/9$	$9\sqrt{9}-\cdot\dot{9}\dot{9}$
27	$9\sqrt{9}+9-9$	$9\sqrt{9}+9-9$
28	$9\sqrt{9}+9/9$	$9\sqrt{9}+\cdot\dot{9}\dot{9}$
29	$9\sqrt{9}+\cdot\dot{9}+\cdot\dot{9}$	$9\sqrt{9}+\cdot\dot{9}+\cdot\dot{9}$
30	$9+9+9+\sqrt{9}$	$9+9+9+\sqrt{9}$
31	$9\sqrt{9}+\sqrt{9}+\cdot\dot{9}$	$9\sqrt{9}+\sqrt{9}+\cdot\dot{9}$
32	$9\sqrt{9}+(\sqrt{9})!-\cdot\dot{9}$	$9\sqrt{9}+(\sqrt{9})!-\cdot\dot{9}$
33	$9\sqrt{9}+9-\sqrt{9}$	$\sqrt{9}(9\cdot\dot{9}+\cdot\dot{9})$

Digitized by Google

n	*Four nines.*	*Four nines.*
34	$9\sqrt{9}+(\sqrt{9})!+\cdot\dot{9}$	$9\sqrt{9}+(\sqrt{9})!+\cdot\dot{9}$
35	$9\sqrt{9}+9-\cdot\dot{9}$	$9\sqrt{9}+9-\cdot\dot{9}$
36	$9+9+9+9$	$9+9+9+9$
37	$9\sqrt{9}+9+\cdot\dot{9}$	$9\sqrt{9}+9+\cdot\dot{9}$
38	$(\sqrt{9})!(\sqrt{9})!+\cdot\dot{9}+\cdot\dot{9}$	$(\sqrt{9})!(\sqrt{9})!+\sqrt{9}-\cdot\dot{9}$
39	$9\sqrt{9}+9+\sqrt{9}$	$9\sqrt{9}+9+\sqrt{9}$
40	$(\sqrt{9})!(\sqrt{9})!+\sqrt{9}+\cdot\dot{9}$	$9\cdot\dot{9}(\sqrt{9}+\cdot\dot{9})$
41	$(\sqrt{9})!(\sqrt{9})!+(\sqrt{9})!-\cdot\dot{9}$	$(\sqrt{9})!(\sqrt{9})!+(\sqrt{9})!-\cdot 9$
42	$9\sqrt{9}+9+(\sqrt{9})!$	$99/\sqrt{9}+9$
43	$(\sqrt{9})!(\sqrt{9})!+(\sqrt{9})!+\cdot\dot{9}$	$(\sqrt{9})!(\sqrt{9})!+(\sqrt{9})!+\cdot\dot{9}$
44	$[(\sqrt{9})!-\cdot\dot{9}]\,9-\cdot\dot{9}$	$(\sqrt{9})!(\sqrt{9})!+9-\cdot\dot{9}$
45	$9+9+9\sqrt{9}$	$9\sqrt{9}+9+9$
46	$[(\sqrt{9})!-\cdot\dot{9}]\,9+\cdot\dot{9}$	$9(\sqrt{9})!-9+\cdot\dot{9}$
47	$9(\sqrt{9})!-(\sqrt{9})!-\cdot\dot{9}$	$9(\sqrt{9})!-(\sqrt{9})!-\cdot\dot{9}$
48	$(9+\sqrt{9})(\sqrt{9}+\cdot\dot{9})$	$9(\sqrt{9})!-\sqrt{9}-\sqrt{9}$
49	$[(\sqrt{9})!+\cdot\dot{9}]\,[(\sqrt{9})!+\cdot\dot{9}]$	$9(\sqrt{9})!-(\sqrt{9})!+\cdot\dot{9}$
50	$9(\sqrt{9})!-\sqrt{9}-\cdot\dot{9}$	$9\cdot\dot{9}\,[(\sqrt{9})!-\cdot\dot{9}]$
51	$(9+9)\sqrt{9}-\sqrt{9}$	$9(\sqrt{9})!-\sqrt{9}.\cdot\dot{9}$
52	$9(\sqrt{9})!-\cdot\dot{9}-\cdot\dot{9}$	$9(\sqrt{9})!-\sqrt{9}+\cdot\dot{9}$
53	$(9+9)\sqrt{9}-\cdot\dot{9}$	$9(\sqrt{9})!-\cdot\dot{9}\dot{9}$
54	$9.9-9\sqrt{9}$	$9\sqrt{9}+9\sqrt{9}$
55	$(9+9)\sqrt{9}+\cdot\dot{9}$	$9(\sqrt{9})!+\cdot\dot{9}\dot{9}$
56	$9(\sqrt{9})!+\cdot\dot{9}+\cdot\dot{9}$	$9(\sqrt{9})!+\cdot\dot{9}+\cdot\dot{9}$
57	$(9+9)\sqrt{9}+\sqrt{9}$	$\sqrt{9}\,(9\cdot\dot{9}+9)$
58	$9(\sqrt{9})!+\sqrt{9}+\cdot\dot{9}$	$9(\sqrt{9})!+\sqrt{9}+\cdot\dot{9}$
59	$(\sqrt{9})!\,(9+\cdot\dot{9})-\cdot\dot{9}$	$9(\sqrt{9})!+(\sqrt{9})!-\cdot\dot{9}$
60	$(9+9)\sqrt{9}+(\sqrt{9})!$	$9\cdot\dot{9}\,(9-\sqrt{9})$
61	$(\sqrt{9})!\,(9+\cdot\dot{9})+\cdot\dot{9}$	$9(\sqrt{9})!+(\sqrt{9})!+\cdot\dot{9}$
62	$9(\sqrt{9})!+9-\cdot\dot{9}$	$9(\sqrt{9})!+9-\cdot\dot{9}$
63	$9.9-9-9$	$9(\sqrt{9})!+9.\cdot\dot{9}$
64	$(9-\cdot\dot{9})(9-\cdot\dot{9})$	$9(\sqrt{9})!+9+\cdot\dot{9}$
65	$(9-\cdot\dot{9})!/(\sqrt{9})!+9$	$(\sqrt{9}+\cdot\dot{9})^{\sqrt{9}}+\cdot\dot{9}$
66	$9.9-9-(\sqrt{9})!$	$9(\sqrt{9})!+9+\sqrt{9}$
67	$(\cdot\dot{9}+\cdot\dot{9})^{(\sqrt{9})!}+\sqrt{9}$	$(\sqrt{9}+\cdot\dot{9})^{\sqrt{9}}+\sqrt{9}$
68	$\Gamma\sqrt{9}\,[(\sqrt{9})!\,(\sqrt{9})!-\Gamma\sqrt{9}]$	$\{[(\sqrt{9})!]!/9\}-9-\sqrt{9}$
69	$9.9-9-\sqrt{9}$	$9.9-9-\sqrt{9}$
70	$[(\sqrt{9})!+\cdot\dot{9}]\,(9+\cdot\dot{9})$	$[(\sqrt{9})!+\cdot\dot{9}]\,(9\cdot\dot{9})$
71	$9.9-9-\cdot\dot{9}$	$9.9-9\cdot\dot{9}$
72	$9\,(9-9/9)$	$9.9-9.\cdot\dot{9}$
73	$9.9-9+\cdot\dot{9}$	$9.9-9+\cdot\dot{9}$
74	$9.9-(\sqrt{9})!-\cdot\dot{9}$	$9.9-(\sqrt{9})!-\cdot\dot{9}$
75	$9.9-\sqrt{9}-\sqrt{9}$	$9.9-\sqrt{9}-\sqrt{9}$
76	$9.9-(\sqrt{9})!+\cdot\dot{9}$	$9.9-(\sqrt{9})!+\cdot\dot{9}$
77	$9.9-\sqrt{9}-\cdot\dot{9}$	$9.9-\sqrt{9}-\cdot\dot{9}$
78	$9.9-9/\sqrt{9}$	$9.9-\sqrt{9}.\cdot\dot{9}$
79	$9.9-\cdot\dot{9}-\cdot\dot{9}$	$9.9-\sqrt{9}+\cdot\dot{9}$
80	$9.9-9/9$	$9.9-\cdot\dot{9}\dot{9}$

Digitized by Google

n	*Four nines.*	*Four nines.*
81	$9.9+9-9$	$9.9+9-9$
82	$9.9+9/9$	$9.9+\cdot\dot{9}\dot{9}$
83	$9.9+\sqrt{9}-\cdot\dot{9}$	$9.9+\cdot\dot{9}+\cdot\dot{9}$
84	$9.9+9/\sqrt{9}$	$9.9+\sqrt{9}.\cdot\dot{9}$
85	$9.9+\sqrt{9}+\cdot\dot{9}$	$9.9+\sqrt{9}+\cdot\dot{9}$
86	$9.9+(\sqrt{9})!-\cdot\dot{9}$	$9.9+(\sqrt{9})!-\cdot\dot{9}$
87	$9.9+\sqrt{9}+\sqrt{9}$	$9.9+\sqrt{9}+\sqrt{9}$
88	$9.9+(\sqrt{9})!+\cdot\dot{9}$	$9.9+9-\cdot\dot{9}$
89	$9.9+9-\cdot\dot{9}$	$99-9\cdot\dot{9}$
90	$9\,(9+9/9)$	$99-9.\cdot\dot{9}$
91	$9.9+9+\cdot\dot{9}$	$99-9+\cdot\dot{9}$
92	$99-(\sqrt{9})!-\cdot\dot{9}$	$99-(\sqrt{9})!-\cdot\dot{9}$
93	$9.9+9+\sqrt{9}$	$99-\sqrt{9}-\sqrt{9}$
94	$99-(\sqrt{9})!+\cdot\dot{9}$	$99-(\sqrt{9})!+\cdot\dot{9}$
95	$99-\sqrt{9}-\cdot\dot{9}$	$99-\sqrt{9}-\cdot\dot{9}$
96	$99-9/\sqrt{9}$	$99.\cdot\dot{9}-\sqrt{9}$
97	$99-\cdot\dot{9}-\cdot\dot{9}$	$99-\cdot\dot{9}-\cdot\dot{9}$
98	$99-9/9$	$99-\cdot\dot{9}\dot{9}$
99	$99+9-9$	$99.\cdot\dot{9}\dot{9}$
100	$99+9/9$	$99\cdot\dot{9}\dot{9}$
101	$99+\cdot\dot{9}+\cdot\dot{9}$	$99\cdot\dot{9}+\cdot\dot{9}$
102	$99+9/\sqrt{9}$	$99.\cdot\dot{9}+\sqrt{9}$
103	$99+\sqrt{9}+\cdot\dot{9}$	$99+\sqrt{9}+\cdot\dot{9}$
104	$99+(\sqrt{9})!+\cdot\dot{9}$	$99+(\sqrt{9})!-9$
105	$99+9-\sqrt{9}$	$99+\sqrt{9}+\sqrt{9}$
106	$99+(\sqrt{9})!+\cdot\dot{9}$	$99+(\sqrt{9})!+\cdot\dot{9}$
107	$99+9-\cdot\dot{9}$	$99+9-\cdot\dot{9}$
108	$9.9+9\sqrt{9}$	$99+9.\cdot\dot{9}$
109	$99+9+\cdot\dot{9}$	$99+9+\cdot\dot{9}$
110	$99.\cdot\dot{9}/\cdot 9$	$(99/\cdot 9).\cdot\dot{9}$
111	$999/9$	$99+9+(\sqrt{9})$
112	$(\sqrt{9})!!/(\sqrt{9})!-9+\cdot\dot{9}$	$[(\sqrt{9})!-\cdot\dot{9}]!-9+\cdot\dot{9}$
113	$99/\cdot 9+\sqrt{9}$	$99/\cdot 9+\sqrt{9}$
114	$99+9+(\sqrt{9})!$	$99+9+(\sqrt{9})!$
115	$(\sqrt{9})!!/(\sqrt{9})!-(\sqrt{9})!+\cdot\dot{9}$	$[(\sqrt{9})!-\cdot\dot{9}]!+\cdot\dot{9}-(\sqrt{9})!$
116	$99/\cdot 9+(\sqrt{9})!$	$[(\sqrt{9})!-\cdot\dot{9}]!-\sqrt{9}-\cdot\dot{9}$
117	$99+9+9$	$9\,(9\cdot\dot{9}+\sqrt{9})$
118	$(\sqrt{9})!!/(\sqrt{9})!-\cdot\dot{9}-\cdot\dot{9}$	$[(\sqrt{9})!-\cdot\dot{9}]!-\sqrt{9}+\cdot\dot{9}$
119	$99/\cdot 9+9$	$99/\cdot\dot{9}+9$
120	$(99+9)/\cdot 9$	$9\cdot\dot{9}\,(9+\sqrt{9})$
121	$(\sqrt{9})!!/(\sqrt{9})!+9/9$	$[(\sqrt{9})!-\dot{9}]!+\cdot\dot{9}\dot{9}$
122	$(\sqrt{9})!!/(\sqrt{9})!+\cdot\dot{9}+\cdot\dot{9}$	$[(\sqrt{9})!-\dot{9}]!+\sqrt{9}-\cdot\dot{9}$
123	$(\sqrt{9})!!/(\sqrt{9})!+9/\sqrt{9}$	$99+(\sqrt{9}+\cdot\dot{9})!$
124	$[(\sqrt{9})!-\cdot\dot{9}]^{\sqrt{9}}-\cdot\dot{9}$	$[(\sqrt{9})!-\dot{9}]!+\sqrt{9}+\cdot\dot{9}$
125	$(\sqrt{9}+\cdot\dot{9}+\cdot\dot{9})^{\sqrt{9}}$	$[(\sqrt{9})!-\dot{9}]!+(\sqrt{9})!-\cdot\dot{9}$
126	$99+9\sqrt{9}$	$99+9\sqrt{9}$
127	$(\sqrt{9})!!/(\sqrt{9})!+(\sqrt{9})!+\cdot\dot{9}$	$[(\sqrt{9})!-\dot{9}]!+(\sqrt{9})!+\cdot\dot{9}$

Digitized by Google

n	*Four nines.*	*Four nines.*
128	$(\sqrt{9})!!/(\sqrt{9})!+9-\cdot\dot{9}$	$[(\sqrt{9})!-\cdot\dot{9}]!+9-\cdot\dot{9}$
129	$(\sqrt{9})!!/(\sqrt{9})!+\sqrt{9}\sqrt{9}$	$[(\sqrt{9})!-\cdot\dot{9}]!+9.\cdot\dot{9}$
130	$(\sqrt{9})!!/(\sqrt{9})!+9+\cdot\dot{9}$	$[(\sqrt{9})!-\cdot\dot{9}]!+9+\cdot\dot{9}$
131	$[(\sqrt{9})!-\cdot\dot{9}]^{\sqrt{9}}+(\sqrt{9})!$	$[(\sqrt{9})!-\cdot\dot{9}]^{\sqrt{9}}+(\sqrt{9})!$
132	$9[9+(\sqrt{9})!]-\sqrt{9}$	$[(\sqrt{9})!-\cdot\dot{9}]!+9+\sqrt{9}$
133	$\Gamma[(\sqrt{9})!]+9+\sqrt{9}+\cdot\dot{9}$	Wanting
134	$9[9+(\sqrt{9})!]-\cdot\dot{9}$	$(\sqrt{9})!!/9+9(\sqrt{9})!$
135	$9(9+\sqrt{9}+\sqrt{9})$	$9(9+\sqrt{9}+\sqrt{9})$
136	$9[9+(\sqrt{9})!]+\cdot\dot{9}$	$(\sqrt{9})!^{\sqrt{9}}-(\sqrt{9})!!/9$
137	$\Gamma[(\sqrt{9})!]+9+9-\cdot\dot{9}$	$9.9+{}_{(\sqrt{9})!}H_{\sqrt{9}}$
138	$9[9+(\sqrt{9})!]+\sqrt{9}$	$[(\sqrt{9})!-\cdot\dot{9}]!+9+9$
139	$\Gamma[(\sqrt{9})!]+9+9+\cdot\dot{9}$	${}_{9}C_{\sqrt{9}}+{}_{\sqrt{9}}H_{9}$
140	$\Gamma[(\sqrt{9})!]+9+9+\Gamma(\sqrt{9})$	$[(\sqrt{9})!+\cdot\dot{9}]!/[(\sqrt{9})!\,(\sqrt{9})!]$

Note on Solution (I.).—In this problem the chief difficulty appears to me to be to find the simplest expression for each number. In this solution an attempt has been made to use the simplest arithmetical and algebraic symbols possible in each instance. Thus, the factorial sign (!) has not been used when simpler symbols sufficed, and the gamma-function symbol has been used only five times (*i.e.* only when its use seemed unavoidable).

[The problem of expressing all numbers up to 200 by three fours was proposed some years ago in the *Royal Engineer Journal*, by Major W. H. Turton, R.E.; his solution was published in that journal, January, 1892.]

15559. (Professor R.W. Genese, M.A.)—An arc PQ of a uniform wire in the shape of an equiangular spiral attracts a particle at the pole O according to the law of nature. Prove that the resultant attraction is proportional to $\dfrac{\text{chord PQ}}{\text{OP}\,.\,\text{OQ}}$.

Solutions (I.) *by* Frances E. Cave; (II.) *by the* Proposer.

(I.) $r = ae^{\theta\cot\alpha}$; therefore, if m be the line density, the component attraction parallel to the initial line

$$= \int_{\theta_1}^{\theta_2} \frac{\cos\theta\, m\, ds}{r^2} = \frac{m}{a\sin\alpha}\int_{\theta_1}^{\theta_2} e^{-\theta\cot\alpha}\cos\theta\, d\theta$$

$$= m\left(\frac{\cos(\theta_1+\alpha)}{r_1} - \frac{\cos(\theta_2+\alpha)}{r_2}\right).$$

The component perpendicular to this

$$= \int_{\theta_1}^{\theta_2} \frac{\sin\theta\, m\, ds}{r^2} = m\left(\frac{\sin(\theta_1+\alpha)}{r_1} - \frac{\sin(\theta_2+\alpha)}{r_2}\right);$$

the resultant $= \dfrac{m\sqrt{[r_1^2+r_2^2-2r_1r_2\cos(\theta_2-\theta_1)]}}{r_1r_2} = m\dfrac{\text{chord PQ}}{\text{OP}\,.\,\text{OQ}}$.

Digitized by Google

(II.) The equation to the curve is $r = ae^{c\theta}$ ($c = \cot\alpha$), and we may consider OP = a as the initial line. To avoid a little complication let us replace attractions by repulsions.

As a vector the force ds/r^2 is represented by

$$\frac{ds}{r^2}\epsilon^{i\theta} = \frac{dr\sec\alpha}{r^2}\epsilon^{i\theta} = \frac{ac\epsilon^{c\theta}d\theta\sec\alpha\epsilon^{i\theta}}{a^2\epsilon^{2c\theta}} = \frac{1}{a\sin\alpha}\epsilon^{(i-c)\theta}d\theta.$$

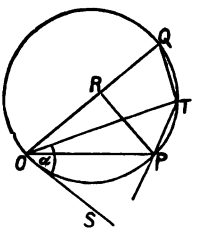
Integrating, the resultant

$$= \frac{1}{a\sin\alpha}\left[\frac{\epsilon^{(i-c)\theta}}{i-c}\right]_0^\theta = \frac{1}{a\sin\alpha}\frac{1-\epsilon^{(i-c)\theta}}{c-i} = \frac{1}{a^2\epsilon^{c\theta}}\frac{ae^{c\theta}-a\epsilon^{i\theta}}{\cos\alpha - i\sin\alpha}$$

$$= \frac{1}{OP.OQ}[OQ - OP(\cos\theta + i\sin\theta)]\epsilon^{i\alpha},$$

or, drawing PR perpendicular to OQ,

$$= \frac{1}{OP.OQ}[RQ - i(RP)]\epsilon^{i\alpha},$$



the tensor of which is PQ/(OP.OQ).

As regards the direction, the factor $RQ - i(RP)$ turns OP through $(-\angle OQP)$, and is, therefore, along the tangent OS to the circle OPQ.

The factor $\epsilon^{i\alpha}$ then turns OS along the chord OT say. Now

$$\angle OQT = \angle SOT = \alpha,$$

and $$\angle OPT = \pi - \alpha.$$

Thus TP, TQ are tangents to the spiral.

15576. (Solidus.)—Given

$$(c+x)(c-y) = (\mu^2-1)(1-c^2), \quad (z-x)(z+y) = (\mu^2-1)(1-z^2);$$

prove that the equations got by interchanging x, y, and μ with c, z, and $1/\mu$ must also hold good.

Solutions (I.) *by* R. F. Davis, M.A.; (II.) *by* C. M. Ross.

(I.) It will be found that the given equations are the results of eliminating z, c from the relations

$$1 - xy = \mu^2(1-cz), \quad (x-y) + \mu^2(c-z) = 0.$$

The interchange stated in the Question will reproduce these same equations.

(II.) The equations got by interchanging the quantities are

$$(x+c)(x-z) = (1/\mu^2-1)(1-x^2), \ (y-c)(y+z) = (1/\mu^2-1)(1-y^2)\ldots(\text{iii., iv.}).$$

From equations (i.) and (ii.) by subtraction, $x(c+z) - y(c+z) = \mu^2(z^2-c^2)$; therefore $y = x - \mu^2(z-c)$. Substituting in equation (i.),

$$\mu^2(xz + cz - cx) = x^2 - 1 + \mu^2;$$

Digitized by Google

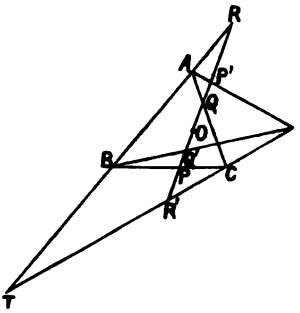
therefore $x^2 - xz - cz + cx = x^2 - x^2/\mu^2 + 1/\mu^2 - 1$:

therefore $(x+c)(x-z) = (1/\mu^2 - 1)(1-x^2)$,

which is the required equation (iii.). Similarly, by eliminating x from equations (i.) and (ii.), the equation (iv.) is obtained.

15584. (James Blaikie, M.A.)—Let O be any point in the plane of a triangle ABC, and let any straight line through O meet BC in P, CA in Q, AB in R; then, if points P′, Q′, R′ be taken on the line, so that PO = OP′, QO = OQ′, RO = OR′, prove that AP′, BQ′, CR′ are concurrent.

Solutions (I.) *by* R. F. Davis, M.A.;
(II.) *by* Professor Nanson.



(I.) Produce CR′ to meet AB in T. Then

$$(RABT) = C(RABT') = (RQPR') = (R'PQR) = (RP'Q'R').$$

Hence AP′, BQ′, TR′ (CR′) are concurrent.

(II.) AP′, BQ′, CR′ are the polars of the collinear points P, Q, R with respect to any hyperbola passing through O, having an asymptote parallel to PQR, and cutting AP, BQ, CR harmonically, and are therefore concurrent.

12211. (Rev. T. R. Terry, M.A.)—If a and b are any positive quantities, find the sum of all the fractions $ab/[b + s(a+b)]$, where s may be zero or any integer, positive or negative.

Solution by Professor Sanjána, M.A.

If s be positive (including zero), the series of fractions is ab/b, $ab/(2b+a)$, $ab/(3b+2a)$, ..., and the sum is

$$ab\int_0^1 (x^{b-1} + x^{2b+a-1} + x^{3b+2a-1} + \ldots)\,dx = ab\int_0^1 \frac{x^{b-1}}{1-x^{a+b}}\,dx.$$

Similarly, if s be negative, the series has its sum

$$= -ab\left(\frac{1}{a} + \frac{1}{2a+b} + \frac{1}{3a+2b} + \ldots\right) = -ab\int_0^1 \frac{x^{a-1}}{1-x^{a+b}}\,dx.$$

Hence the whole sum is $\displaystyle ab\int_0^1 \frac{x^{b-1} - x^{a-1}}{1-x^{a+b}}\,dx,$

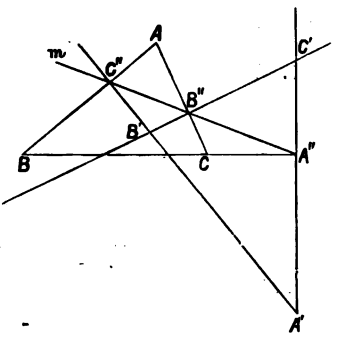
and may be evaluated by the integral calculus.

Digitized by Google

12264. (Professor NEUBERG.) — Une droite m coupe les côtés d'un triangle ABC en trois points A″, B″, C″ tels que les perpendiculaires élevées en ces points sur BC, CA, AB forment un triangle A′B′C′ de surface donné. Trouver l'enveloppe de la droite m.

Solution by W. J. GREENSTREET, M.A.

Since BA, B′A′, ..., intersect on a line m, the joins of corresponding vertices must be concurrent in a point M. As m moves parallel to itself, M remains the centre of similitude for all triangles A′B′C′, and when one of these triangles degenerates into M, m is then a Simson-line. Hence M is on the circle ABC, and its Simson-line for ABC is parallel to m. Also, as ABC is deducible from A′B′C′ and m, just as A′B′C′ is deduced from ABC and m, it follows that M is on the circle A′B′C′, and its Simson-line for A′B′C′ is parallel to m. Since ABC, A′B′C′ have their homologous sides respectively at right angles, then, if O, O′ are the centres of their circum-circles, we see that OA and O′A′ are at right angles; and, further, OMO′ is a right angle. As m moves, A′B′C′ being of constant area, MO′ is of constant length, and the locus of O′ is a circle concentric with the circum-circle ABC. Then, since O′A′ moves parallel to itself and remains constant in magnitude, the locus of A′ is a circle, centre O″, where OO″ is equal and parallel to O′A′. Finally, the right-angled triangles OO″A and O′MO are equal; therefore O″A = OO′ = O″A′; therefore the locus of A′ passes through A. Hence m is the Simson-line of a point A′ on the circle O″ with reference to the inscribed angle ABC. Therefore it envelops a three-cusped hypocycloid.



Note by the PROPOSER.

Au sujet des triangles orthohomologiques ABC, A′B′C′, on peut voir *Mathesis*, 1886, p. 116; 1895, pp. 265–8; 1896, p. 81. La droite A″B″C″ passe au milieu de la droite joignant les orthocentres des triangles ABC, A′B′C′; le centre de similitude dè ces triangles est le second point de rencontre des circonférences circonscrites.

15499. (Lt.-Col. ALLAN CUNNINGHAM, R.E. — *Revised.*) — Let $N = 111\ldots11$ consist of q^n units in the scale of $r = mq + 1$, q being *prime;* then $N = Mq^n$.

Digitized by Google

Solutions (I.) *by* JAMES BLAIKIE, M.A., *and* R. CHARTRES; (II.) *by the* PROPOSER.

(I.) We have $N = 111\ldots1$, q^n factors. Let $p = q^n$. Then

$$N = \frac{r^p-1}{r-1} = \frac{(1+mq)^p-1}{mq} = \frac{1}{mq}\left(pmq + \frac{p(p-1)}{1.2}m^2q^2 + \ldots\right).$$

It has to be shown that each term of this expansion contains pq or q^{n+1} as a factor. The general term is $\frac{p(p-1)\ldots(p-x+1)}{1.2\ldots r}m^xq^x$, which clearly contains pq as a factor, when x is not a multiple of q. Consider the case when x is a power of q, or a multiple of a power of q. Let $x = aq^t$. Then the general term contains the factor q^{n-t+x}, which cannot be less than q^{n+1}, unless $t+1>x$, or $t+1>aq^t$, which is impossible, since $q>1$. Thus $N = Mq^n$.

(II.) In order to avoid heavy double exponents, let R_x denote r_{q^x}, so that $R_2 = r^{q^2}$, $R_1 = r^q$, $R_0 = r$; thus $R_{x+1} = r^{q^{x+1}} = (r^{q^x})^q = R_x^q$. Here

$$N = 111\ldots1 \ [q^n \text{ units}] = (r^{q^n}-1)/(r-1) - (R_n-1)/(R_0-1).$$

Now let N be resolved into its n *algebraic* prime factors, *i.e.*,

$$N = \frac{R_n-1}{R_{n-1}-1}\cdot\frac{R_{n-1}-1}{R_{n-2}-1}\cdots\frac{R_2-1}{R_1-1}\cdot\frac{R_1-1}{R_0-1}.$$

It will be shown that each one of these n fractions contains q. Let

$$F_x = \frac{R_{x+1}-1}{R_x-1} = \frac{R_x^q-1}{R_x-1} = R_x^{q-1} + R_x^{q-2} + \ldots + R_x^2 + R_x + 1 \ [q \text{ terms}].$$

Take the term $R_x^t = (mq+1)^{q^t} = Mq+1$, where Mq means "multiple of q"; therefore $F_x = \Sigma(Mq+1) = Mq+q$ units $= Mq$. A similar argument applies to every fraction F_x, and N contains n such fractions; therefore N contains q^n.

A similar argument to the above would show that, if $N = 100\ldots01$, with (q^n-1) ciphers, in the scale of $r = mq-1$, then $N = Mq^n$. The argument shows that the limitation q = prime (stated in the Question) is unnecessary.

15553. (Professor NANSON.)—Eliminate x, y, z from

$$ax^2 + by^2 + cz^2 = 0,$$

$$(b-c)\,lyz + (c-a)\,mzx + (a-b)\,nxy = 0, \quad lx + my + nz = 0.$$

Solutions (I.) *by* THOMAS MUIR, LL.D., F.R.S.; (II.) *by* A. M. NESBITT, M.A., *and others.*

(I.) Probably the most direct mode of solution is first to eliminate z, and then from the two resulting equations to eliminate x and y. These equations are

$$(an^2 + cl^2)\,x^2 + 2clmxy + (bn^2 + cm^2)\,y^2 = 0,$$

$$Blnx^2 + (Al^2 + Bm^2 - Cn^2)\,xy + Almy^2 = 0,$$

where A, B, C are written for $b-c$, $c-a$, $a-b$ respectively. Their eliminant is of the twelfth degree, a factor of it, n^2, being irrelevant to the original set of equations.

Digitized by Google

It is of greater interest, however, to observe the result of applying as follows the dialytic method :—From the linear equation three quadrics arise through multiplication by x, y, z: there thus is needed only one other such equation in order to have a complete set of six from which to eliminate x^2, y^2, z^2, yz, zx, xy. This want is supplied by taking the given equations in the form

$$ax \,.\, x + by \,.\, y + cz \,.\, z = 0, \qquad \mathrm{C}ny \,.\, x + \mathrm{A}lz \,.\, y + \mathrm{B}mx \,.\, z = 0,$$
$$l \,.\, x + m \,.\, y + n \,.\, z = 0,$$

and thus deriving by elimination

$$\begin{vmatrix} ax & by & cz \\ \mathrm{C}ny & \mathrm{A}lz & \mathrm{B}mx \\ l & m & n \end{vmatrix} = 0,$$

i.e., $\quad a\mathrm{B}m^2x^2 + b\mathrm{C}n^2y^2 + c\mathrm{A}l^2z^2 - c\mathrm{C}mnyz - a\mathrm{A}nlzx - b\mathrm{B}lmxy = 0.$

The eliminant as a six-line determinant thus is

$$\begin{vmatrix} a & b & c & . & . & . \\ . & . & . & \mathrm{A}l & \mathrm{B}m & \mathrm{C}n \\ l & . & . & . & n & m \\ . & m & . & n & . & l \\ . & . & n & m & l & . \\ a\mathrm{B}m^2 & b\mathrm{C}n^2 & c\mathrm{A}l^2 & -c\mathrm{C}mn & -a\mathrm{A}nl & -b\mathrm{B}lm \end{vmatrix}.$$

This is seen to separate into six parts arrangeable in two sets of three, viz. :—

$$\left\{ b\mathrm{B}lm \begin{vmatrix} a & b & c & . & . \\ . & . & . & \mathrm{A}l & \mathrm{B}m \\ l & . & . & . & n \\ . & m & . & n & . \\ . & . & n & m & l \end{vmatrix} \right\} - \Sigma \left\{ \mathrm{A}^2 \begin{vmatrix} a & b & . & . & . \\ . & . & \mathrm{A}l & \mathrm{B}n & \mathrm{C}m \\ l & . & . & n & m \\ . & m & n & . & l \\ . & . & m & l & . \end{vmatrix} \right\};$$

so that the non-determinant form is readily calculable and found to be $\Sigma bc\mathrm{A}^2l^6 + \Sigma ca\mathrm{A}^2l^4m^2 + \Sigma bc\mathrm{B}^2l^2m^4 - 2\Sigma bc\mathrm{AB}l^4m^2 - 2\Sigma ca\mathrm{AB}l^2m^4 + \Sigma a^2\mathrm{A}^2l^2m^2n^2$. But the sum of the first three terms here is the product of

$$bcl^2 + cam^2 + abn^2 \quad \text{and} \quad \mathrm{A}^2l^4 + \mathrm{B}^2m^4 + \mathrm{C}^2n^4 \;;$$

and, if we substitute for $\Sigma a^2\mathrm{A}^2$ its equivalent $-2\Sigma bc\mathrm{BC}$, the sum of the last three terms is with equal ease seen to be the product of

$$bcl^2 + cam^2 + abn^2 \quad \text{and} \quad -2\mathrm{BC}m^2n^2 - 2\mathrm{CA}n^2l^2 - 2\mathrm{AB}l^2m^2.$$

Thus our final result is $\Sigma bcl^2\,(\Sigma \mathrm{A}^2l^4 - 2\Sigma \mathrm{BC}m^2n^2)$.

(II.) If we write $l(b-c) = \lambda$, $m(c-a) = \mu$, $n(a-b) = \nu$, we have to eliminate x, y, z from

$$\mathrm{S} \equiv ax^2 + by^2 + cz^2 = 0, \quad \mathrm{S}' \equiv \lambda yz + \mu zx + \nu xy = 0, \quad \mathrm{L} \equiv lx + my + nz = 0.$$

The eliminant is, of course, $\Phi^2 = 4\Sigma\Sigma'$. Now

$$\Sigma = bcl^2 + cam^2 + abn^2,$$
$$\Sigma' = -l^2\lambda^2 - m^2\mu^2 - n^2\nu^2 + 2mn\mu\nu + \ldots + \ldots,$$
$$\Phi = -2\,(a\lambda mn + b\mu nl + c\nu lm);$$

so that Φ is identically zero.

L is, in fact, cut harmonically by S and S′, and consequently the three loci can only have a common point provided L touches either S or S′.

Digitized by Google

Thus the eliminant is $\Sigma\Sigma' = 0$. The same thing comes also easily from elementary considerations as follows:—From (2) it is clear that we may put $lyz = a\theta + \phi$, $mzx = b\theta + \phi$, $nxy = c\theta + \phi$, which at once give

$$\phi\,(x^2 + y^2 + z^2) = 0.$$

Hence either (1) $\phi = 0$, whence $yz = a\theta/l$, ...; so that $a^2x = lbc/mn$, yielding $l^2bc + m^2ca + n^2ab = 0$, which is $\Sigma = 0$; or else $x^2 + y^2 + z^2 = 0$, which, taken in conjunction with $ax^2 + by^2 + cz^2 = 0$, gives

$$x^2/(b-c) = y^2/(c-a) = z^2/(a-b);$$

so that $$l\sqrt{(b-c)} \pm m\sqrt{(c-a)} \pm n\sqrt{(a-b)} = 0,$$

i.e., $$l^4(b-c)^2 + \ldots + \ldots - 2m^2n^2(c-a)(a-b) - \ldots - \ldots = 0,$$

which is $\Sigma' = 0$.

15547. (Professor Sanjána, M.A.)—A central conic, whose minor axis is of given length, touches two fixed straight lines; if one focus moves on a given straight line, prove that the other moves on a hyperbola having its asymptotes parallel to the fixed lines.

Solutions (I.) *by* Frances E. Cave; (II.) *by* A. R. Richardson.

(I.) Take the two fixed lines as axes. The product of the perpendiculars from the foci on to either axis is constant, and therefore the products of the abscissæ and of the ordinates are constant. Therefore, if one focus moves along $lx + my + n = 0$, the other moves along the hyperbola $l/x + m/y + n/k^2 = 0$.

(II.) Let $x = 0$, $y = mx$ be the equations to the two tangents; b the semi-minor axis; $\xi x + y = C$ the locus of one of the foci. Then we know that the product of distances of foci from any tangent is b^2; therefore

$$Xy' = b^2, \quad (Y - mX)(y' - mx') = b^2(1 + m^2),$$

where (X, Y) are the points we seek, and (x', y') of the other focus.

Eliminating x', y' from these two equations and $\xi x' + y' = C$, we get

$$(Y - mX)\,[b^2(1 + m/\xi) - XC(m/\xi)] = b^2\,[X(1 + m^2)],$$

i.e., a hyperbola; and, as asymptotes are parallel to terms of the second degree, they are parallel to $X = 0$, $Y = mX$.

15555. (Robt. W. D. Christie.)—Let $p = a^2 + b^2$, $q = ac + bd$, $s = c^2 + d^2$, $ad - bc = 1$: prove $r^{2m} + 1 = ps$ and $r^{4m} + 1 = p(p - 2q + s)$; and more generally $(pt + r^m)^2 + 1 = pS$, $(pt + r^{2m})^2 + 1 = pT$, where r is any primitive root of prime $4m + 1$, t any integer. Establish similar equations for prime $p = 4m - 1$.

Digitized by Google

Solution by the PROPOSER.

(A) By adaptation of Euler's theorem we have

$$(ac+bd)(a^2+b^2-ac-bd)-(a^2+b^2)(ac+bd-c^2-d^2) = (ad-bc)^2 = 1 \text{...(i.).}$$

Adapting to the prime roots, we get

$$r^m(p-r^m)-p[(p^2-r^{6m}-r^{2m}-2)/2p] = 1 \text{ .....(ii.).}$$

Now $(ac+bd)^2+(ad-bc)^2 = (a^2+b^2)(c^2+d^2)$,

i.e., $(r^m)^2+1 = ps$ .....(1),

$$(r^{3m})^2+1 = (p-r_m)^2+1 = p^2-2pr^m+r^{2m}+1 = p^2-2pq+ps$$
$$= p(p-2q+s) \text{ .....(2),}$$

(3) and (4) follow immediately from (1) and (2).

(B) For primes $= 4m-1$, we have, corresponding with (i.) and (ii.) *supra*,

$$(rac+bd)(ra^2+b^2-rac-bd)-(ra^2+b^2)(rac+bd-rc^2-d^2)$$
$$= r(ad-bc)^2 = r \text{.....(iii.)}$$

when $ad-bc = 1$, $r^{m+1}.r^{3m+2}-pq = r$ .....(iv.).

15216. (R. C. ARCHIBALD, M.A., Ph.D.)—The *trisectrix* of Maclaurin ($r\cos\frac{1}{3}\phi = C$ or $r\sin 2\theta + a\sin 3\theta = 0$), the *logocyclic* curve or *strophoide* ($r\cos\theta + a\cos 2\theta = 0$), and the *cubic* of Tschirnhausen ($r^3\cos\frac{1}{3}\theta = a^3$) are nodal cubics. Express their equations in the form $X^3+Y^3+Z^3 = 0$, where X, Y, Z are linear functions of the Cartesian co-ordinates.

Additional Solution by Professor SIRCOM.

In each of these curves there is but one real point of inflexion; hence two of the expressions X, Y, Z must be imaginary. The algebraical form of the first equation is

$$2x(x^2+y^2)+a(3x^2-y^2) = 0.$$

This may be written $a^3 = (a-2x)(x^2+y^2+2ax+a^2)$;

whence $a^3 = (a-2x)(x+\iota y+a)(x-\iota y+a)$,

which is the rationalized form of

$$(x+\iota y+a)^{\frac{1}{3}}+(x-\iota y+a)^{\frac{1}{3}}+(a-2x)^{\frac{1}{3}} = 0.$$

The algebraical form of the second is

$$x(x^2+y^2)+a(x^2-y^2) = 0.$$

The asymptote $a-x = 0$ is inflexional: assume, therefore,

$$(lx+m\iota y+a)^{\frac{1}{3}}+(lx-m\iota y+a)^{\frac{1}{3}}+(a-x)^{\frac{1}{3}} = 0.$$

Rationalizing and equating coefficients, we have for l, m

$$(2l-1)^3-27l^2 = 9(l+1)^2 = 27m^2,$$

which give $l = \frac{1}{4}$, $m = \frac{3}{4}\sqrt{3}$, and then

$$(5x+3\sqrt{3}\iota y+4a)^{\frac{1}{3}}+(5x-3\sqrt{3}\iota y+4a)^{\frac{1}{3}}+(4a-4x)^{\frac{1}{3}} = 0.$$

The third is $(4a-x)^3 = 27a(x^2+y^2)$.

This can be written $(2x-8a)^3 = -27.8a(x^2+y^2)$,

which is the rationalized form of

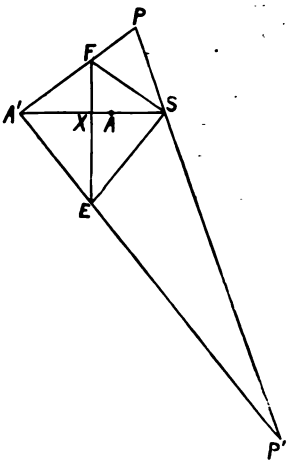
$$(x+\iota y)^{\frac{1}{3}}+(x-\iota y)^{\frac{1}{3}}-2a^{\frac{1}{3}} = 0.$$

Digitized by Google

15563. (R. F. Davis, M.A.)—Given a hyperbola whose eccentricity = 2, prove that the circle described upon any focal chord as diameter passes through the further extremity of the transverse axis.

Geometrical Solutions (I.) *by* K. Deva Rau, B.A., *and* W. F. Beard, M.A.; (II.) *by* Professor Sanjána, M.A.

(I.) Let PSP′ be the focal chord, AA′ the transverse axis, and EXF the directrix. Join A′P and A′P′, cutting the directrix in F and E. Join FS and ES. Then FS bisects the angle A′SP, and ES bisects the angle A′SP′; therefore ESF is a right angle.



Again, the eccentricity of the hyperbola = 2; therefore X is the middle point of SA′. Also EXF is perpendicular to SA′; therefore

$$\angle FA'X = \angle FSX$$

and $$\angle EA'X = \angle ESX;$$

therefore

$$\angle EA'F = \angle ESF = 90°.$$

Therefore the circle described on PSP′ as diameter passes through A′.

(II.) Since $e = 2$,

$$BC^2 = 3CA^2,$$

$$CS = 2CA, \quad CX = \tfrac{1}{2}CA;$$

also $$PM^2 = 3AM \cdot A'M,$$

$$QN^2 = 3AN \cdot A'N.$$

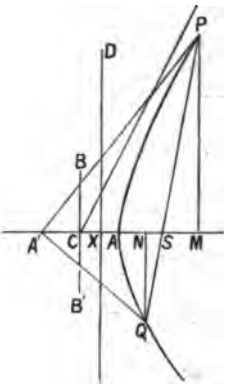
Now $$SP = eXM = 2(A'M - A'X)$$

$$= 2A'M - 3a,$$

and $$SQ = 2A'N - 3a;$$

so that

$$\frac{2A'M - 3a}{2A'N - 3a} = \frac{SM}{SN} = \frac{A'M - 3a}{3a - A'N};$$



therefore

$$\frac{3A'M - 6a}{A'N} = \frac{A'M}{3A'N - 6a};$$

whence

$$9(A'M - 2a)(A'N - 2a) = A'N \cdot A'M,$$

or $$A'M \cdot A'N = 9AM \cdot AN.$$

Combining, we get $PM^2 \cdot QN^2 = 9AN \cdot AM \cdot A'N \cdot A'M = (A'N \cdot A'M)^2$; therefore $PM \cdot QN = A'N \cdot A'M$, *i.e.*, $PM/A'M = A'N/QN$. Thus the triangles PA′M, QA′N are similar, and

$$\angle PA'M = \angle NQA' = 90° - \angle QA'N;$$

hence $\angle PA'Q = 90°$; so that A′ lies on the circle described on PQ as diameter.

Digitized by Google

15604. (Professor LAUVERNAY.)—Construire un triangle rectangle connaissant en position une droite Δ sur laquelle se trouve l'hypoténuse, le centre Ω du cercle enveloppant les cercles ex-inscrits au triangle ainsi que son rayon ρ.

Solution by Professor SANJÁNA, M.A.

The circle enveloping the escribed circles is the nine-point circle, and the centre of this is the mid-point of the median to the hypotenuse. Hence, with Ω as centre and ρ as radius, describe a circle, cutting the right line Δ in M; draw the diameter MΩA, as also the diameter LΩN parallel to Δ; join AL, AN, and produce these to meet Δ in B and C. Then ABC is the required right-angled triangle.

As the circle cuts Δ in two points (in general), there are two triangles given by the solution, which are images of one another with regard to the perpendicular from Ω on Δ.

15536. (*Communicated by* J. L. S. HATTON.)—Evaluate the determinant

$$\begin{vmatrix} a & b & c & d & e & f & g & h \\ h & a & b & c & d & e & f & g \\ g & h & a & b & c & d & e & f \\ f & g & h & a & b & c & d & e \\ h & g & f & e & d & c & b & a \\ a & h & g & f & e & d & c & b \\ b & a & h & g & f & e & d & c \\ c & b & a & h & g & f & e & d \end{vmatrix}.$$

Solution by C. M. ROSS.

Taking ω_r to be any root of the equation $x^8-1=0$, and adding to the first column the sum of the constituents of the remaining columns multiplied by $\omega_r, \omega_r^2, \omega_r^3, \omega_r^4, \omega_r^5, \omega_r^6, \omega_r^7$, the result is an equivalent determinant in which the first column is

$$a+\omega_r b+\ldots+\omega_r^7 h \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(i.)},$$
$$h+\omega_r a+\ldots+\omega_r^7 g \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(ii.)},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$b+\omega_r c+\ldots+\omega_r^7 a \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(viii.)}.$$

But the expression (ii.) equals (i.) multiplied by ω_r, (iii.) equals (i.) by ω_r^2, and so on.

Hence (i.) is a factor of the determinant, and, as the product of 8 factors each similar to (i.) contains the term $+a^8$, the value of the given circulant is $\prod_{r=1}^{r=8}(a+\omega_r b+\ldots+\omega_r^7 h)$.

15589. (R. CHARTRES.)—Express $m^n/(mr-1)$ as a radix fraction in scale r.

Digitized by Google

Solutions (I.) *by* Rev. T. Wiggins, B.A.; (II.) *by the* Proposer.

(I.) $$\frac{m^n}{mr-1} = \frac{1}{r}\,\frac{m^n r - m^{n-1} + m^{n-1}}{mr-1} = \frac{m^{n-1}}{r} + \frac{1}{r}\,\frac{m^{n-1}}{mr-1}$$
$$= \frac{m^{n-1}}{r} + \frac{m^{n-2}}{r^2} + \frac{m^{n-3}}{r^3} + \ldots + \frac{m}{r^{n-1}} + \frac{1}{r^n}.$$

(II.) Since $1/(mr-1) = .\dot{a}bc \ldots m\dot{1}$ in scale radix r, therefore
$$m/(mr-1) = .\dot{1}abc \ldots \dot{m}.$$

Hence $m^n/(mr-1)$ can be found by putting the last n figures first and prefixing the integral part, if any. Therefore $1/x \equiv 1/(pq-1)$ will in scales p and q (1) have unity for last figure; (2) multipliers q and p respectively; (3) the same number of recurring figures in scales $\left(mx + \frac{p}{q}\right)$, m = any positive integer; (4) $p^n - 1$ and $q^n - 1$ will be divisible by $pq-1$, n being the number of recurring figures; and (5), if $pq-1$ be a prime number and $n = pq-2$, we have Fermat's theorem.

9811. If masses P, Q, and R be placed at the vertices A, B, and C respectively of the triangle of reference, show that the trilinear equations to the principal axes at the centre of gravity of the masses will be

$$x + my + nz = 0 \quad \text{and} \quad l'x + m'y + n'z = 0,$$

where $l : m : n$ and $l' : m' : n'$ are determined by the equations

$$P(l/a) + Q(m/b) + R(n/c) = 0, \quad P(l'/a) + Q(m'/b) + R(n'/c) = 0,$$
$$P(ll'/a^2) + Q(mm'/b^2) + R(nn'/c^2) = 0,$$

and

$$ll' + mm' + nn' - (mn' + m'n)\cos A - (nl' + n'l)\cos B - (lm' + l'm)\cos C = 0,$$

and that at any point the principal axes are conjugate (*i.e.*, each passes through the pole of the other) with respect to the conics

$$Pl^2/a^2 + Qm^2/b^2 + Rn^2/c^2 = 0$$

and
$$l^2 + m^2 + n^2 - 2mn\cos A - 2nl\cos B - 2lm\cos C = 0$$

(the circular points at infinity).

Solution by A. M. Nesbitt, M.A.

If $(\bar{\alpha}, \bar{\beta}, \bar{\gamma})$ represent the actual values of the co-ordinates of the centre of mass, we have $\bar{\alpha}\Sigma P = P(2\Delta/a), \ldots$; so that $\bar{\alpha} : \bar{\beta} : \bar{\gamma} = P/a : Q/b : R/c$. Hence, since this point lies on both the given lines, we get the first two relations in the Question.

Further, the perpendiculars from A to the two given lines are $(l\,2\Delta/a)/[\phi(lmn)]$ and $(l'\,2\Delta/a)/[\phi(l'm'n')]$, where

$$[\phi(l, m, n)]^2 = \Sigma l^2 - 2\Sigma mn\cos A.$$

Expressing that the products of inertia vanish, we have

$$Pll'/a^2 + Qmm'/b^2 + Rnn'/c^2 = 0,$$

Digitized by Google

which is the third relation to be established, and this applies whether the given lines intersect in the centre of mass or not. The fourth merely ensures that the two given straight lines are at right angles.

Further, the trilinear equation to the conic given tangentially by $P\lambda^2/a^2 + \ldots + \ldots = 0$ is $QRx^2/(b^2c^2) + \ldots + \ldots = 0$, and the condition that (l, m, n), (l', m', n') should be conjugate with respect to this is

$$\begin{vmatrix} QR/(b^2c^2) & . & . & l \\ . & RP/(c^2a^2) & . & m \\ . & . & PQ/(a^2b^2) & n \\ l' & m' & n' & . \end{vmatrix} = 0,$$

and, if we respectively multiply the four rows by $P\lambda'/a^2$, Qm'/b^2, Rn'/c^2, and $-(PQR/a^2b^2c^2)$, their sum is zero in every column; so that the relation is satisfied.

The final part of the theorem is of course evident *a priori*; or it may be established thus. The tangential equation $\Sigma l^2 - 2\Sigma mn \cos A = 0$, when translated to trilinears, becomes $\Sigma x^2 \sin^2 A + 2\Sigma yz \sin B \sin C = 0$, being, in fact, $(\Sigma x \sin A)^2 = 0$; and the condition that (l, m, n), (l', m', n') should be conjugate to this is

$$\begin{vmatrix} \sin^2 A & \sin A \sin B & \sin A \sin C & l \\ \sin A \sin B & \sin^2 B & \sin B \sin C & m \\ \sin A \sin C & \sin B \sin C & \sin^2 C & n \\ l' & m' & n' & \end{vmatrix} = 0.$$

Multiplying the first three rows respectively by $l' - m' \cos C - n' \cos B$, $m' - n' \cos A - l' \cos C$, $n' - l' \cos B - m' \cos A$ and adding, we get zero in every column. Hence the relation is satisfied.

10890. (E. Cesaro.)—Démontrer que la chaînette d'égale résistance, qui a pour équation intrinsèque $\rho = \frac{1}{2}a\,(e^{s/a} + e^{-s/a})$, est la seule ligne dont le rayon de courbure soit égal au segment de la normale intercepté par deux droites fixes parallèles.

Solution by Frances E. Cave.

Since radius of curvature = segment of normal intercepted between two parallel fixed straight lines, therefore its projection on their perpendicular distance is constant; therefore $\rho \cos \psi$ = constant = a, say, where ψ is the inclination of the tangent to the fixed lines.

Put $\sec \psi = \cosh u$; then $ds = a\,du$; therefore $u = s/a$, if s be measured from the point where ψ vanishes; therefore

$$\rho = a \cosh s/a = \tfrac{1}{2}a\,(e^{s/a} + e^{-s/a}).$$

15080 & 15368. (Professor Nanson.)—(15080.) Lines drawn through the vertices of a triangle Δ divide the angles into the segments α, α'

Digitized by Google

β, β'; γ, γ', and form a triangle Δ'. If R, R′ are the radii of the circum-circles of Δ, Δ', prove that

$$\frac{R'}{R} = \frac{\sin\alpha \sin\beta \sin\gamma - \sin\alpha' \sin\beta' \sin\gamma'}{\sin(\beta+\gamma')\sin(\gamma+\alpha')\sin(\alpha+\beta')},$$

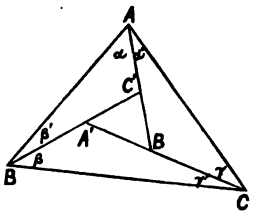
and deduce the theorems of Ceva and Menelaus.

(15368.) If lines drawn through the vertices of a triangle Δ divide the angles into segments α, α'; β, β'; γ, γ', and form a triangle of area Δ', then

$$\frac{\Delta'}{\Delta} = \frac{(\sin\alpha \sin\beta \sin\gamma - \sin\alpha' \sin\beta' \sin\gamma')^2}{\sin(\beta+\gamma')\sin(\gamma+\alpha')\sin(\alpha+\beta')\sin(\alpha+\alpha')\sin(\beta+\beta')\sin(\gamma+\gamma')}.$$

Solutions by the Proposer.

Let $AB' = x$, $AC' = z'$, ..., so that $\alpha' = x - z'$, ...; then, expressing that the large triangle is the sum of the four small triangles, we have



$$\frac{\Delta}{\Delta'} = 1 + \frac{y'z}{b'c'} + \frac{z'x}{c'a'} + \frac{x'y}{a'b'} = \frac{xyz - x'y'z'}{a'b'c'}.$$

Now $z : y' : a = \sin\beta : \sin\gamma' : \sin A'$, ...;
therefore

$$xyz : x'y'z' : abc$$
$$= \Pi \sin\alpha : \Pi \sin\alpha' : \Pi \sin A';$$

therefore

$$\frac{R'}{R} = \frac{a'b'c'\Delta}{abc\Delta'} = \frac{\Pi \sin\alpha - \Pi \sin\alpha'}{\Pi \sin A'} \quad \ldots\ldots\ldots\ldots(1),$$

which gives the formula in Question 15080.

Again $\Delta : \Delta' = bc \sin A : b'c' \sin A' = \ldots$. Hence, from (1) we get

$$\Delta'/\Delta = (\Pi \sin\alpha - \Pi \sin\alpha')^2/(\Pi \sin A \, \Pi \sin A'),$$

which gives the formula in Question 15368.

Further, we have $\Pi \sin\alpha = \Pi \sin\alpha'$ or $xyz = x'y'z'$, when $R' = 0$ or when $R = \infty$, *i.e.*, when the lines through A, B, C are concurrent, or when A, B, C are collinear; and thus we have the theorems of Ceva and Menelaus.

For other solutions see *Reprint*, New Series, Vol. III., p. 74; Vol. V., p. 48.

15577. (Professor H. Langhorne Orchard, M.A., B.Sc.) — Find (1) the sum of n terms of the series

$$0 + 128 + 4374 + 49152 + 312500 + 1399680 + \ldots;$$

and (2) the sum of n terms of the series in (1) added to n terms of the series $1 + 128 + 2187 + 16384 + 78125 + 279936 + \ldots$.

Solution by C. M. Ross; Rev. T. Wiggins, B.A.; *and* K. Deva Rau, B.A.

(1) Series may be written

$$0 \cdot 1^7 + 1 \cdot 2^7 + 2 \cdot 3^7 + 3 \cdot 4^7 + 4 \cdot 5^7 + 5 \cdot 6^7 + \ldots + (n-1)\, n^7;$$

Digitized by Google

$$\therefore \quad S = \Sigma n^8 - \Sigma n^7$$

$$= \tfrac{1}{9}n^9 + \tfrac{1}{2}n^8 + B_1\frac{8n^7}{2!} - B_3\frac{8.7.6n^5}{4!} + B_5\frac{8.7.6.5.4n^3}{6!} - B_7\frac{8.7.6.5.4.3.2n}{8!} - \left(\tfrac{1}{9}n^9 + \tfrac{1}{2}n^7 + B_1\frac{7n^6}{2!} - B_3\frac{7.6.5n^4}{4!} + B_5\frac{7.6.5.4.3n^2}{6!}\right) + C$$

$$= \tfrac{1}{9}n^9 + \tfrac{2}{3}n^8 + \tfrac{1}{6}n^7 - \tfrac{7}{15}n^6 - \tfrac{7}{15}n^5 + \tfrac{7}{24}n^4 + \tfrac{2}{9}n^3 - \tfrac{1}{15}n^2 - \tfrac{1}{30}n$$

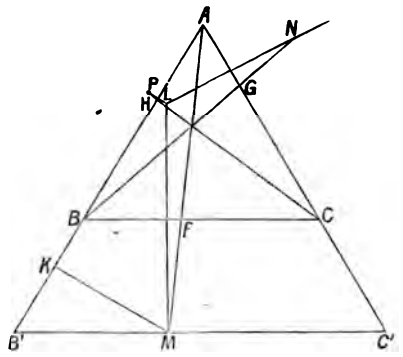
(constant being zero).

(2) $S = \Sigma n^8 - \Sigma n^7 + \Sigma n^7 = \Sigma n^8 = \tfrac{1}{9}n^9 + \tfrac{1}{2}n^8 + \tfrac{2}{3}n^7 - \tfrac{7}{15}n^5 + \tfrac{2}{9}n^3 - \tfrac{1}{30}n.$

N.B.—B_1, B_3, B_5, B_7 are Bernoulli's first four numbers.

10349. (Professor Hain.)—Étant donné un triangle équilatéral ABC, si M, N, P sont les symétriques d'un point quelconque L par rapport aux trois côtés, démontrer que les droites AM, BN, CP se recontrent en un même point.

Solution by James Blaikie, M.A.



Let AM, BN, CP meet BC, CA, AB in F, G, H. Through M draw B'C' parallel to BC, meeting AB, AC produced in B', C'. Draw MK perpendicular to BC. Then

$$MK = \tfrac{1}{2}LP + LM\cos 60° = \tfrac{1}{2}(LP + LM),$$

and

$$B'M = MK \sec 30° = (LP + LM)/\sqrt{3}.$$

Similarly,

$$MC' = (LM + LN)/\sqrt{3};$$

therefore

$$BF/FC = B'M/MC' = (LP + LM)/(LM + LN).$$

Similarly,

$$CG/GA = (LM + LN)/(LN + LP),$$

$$AH/HB = (LN + LP)/(LP + LM);$$

therefore

$$(BF \cdot CG \cdot AH)/(FC \cdot GA \cdot HB) = 1,$$

and AM, BN, CP are concurrent.

1507. (The late Professor Clifford, F.R.S.)—Consider six planes A, B, C, D, E, F and join the point ABC to the point DEF, and so on; we have thus ten finite straight lines and their middle points lie in a plane.

Digitized by Google

Note by Professor Nanson.

When the six planes are $x=0$, $y=0$, $z=0$, $\alpha \equiv y+z-2=0$, $\beta \equiv z+x-2=0$, $\gamma \equiv x+y-2=0$ the theorem is not true. For the co-ordinates of the points $zx\alpha$, $y\beta z$ are (0, 2, 0), (2, 0, 0). The middle point of the join is therefore (1, 1, 0). This middle point and the middle points of the joins $xy\beta$, $zy\alpha$; $yz\gamma$ $x\alpha y$ lie on the plane

$$x+y+z=2.$$

But this does not pass through the middle point (1, 1, 1) of the join xyz, $\alpha\beta\gamma$.

15557. (James Blaikie, M.A.)—If n be a number of r digits such that its square ends in the same r digits as the number, prove that (*a*) 10^r+1-n has the same property; (*b*) the square of $n-1$ ends in the same r digits as 10^r+1-n; (*c*) the cube of $(n-1)$ ends in the same r digits as $n-1$. Also, if n be a number of r digits such that its cube ends in the same r digits, prove that (*d*) 10^r-n has the same property. Examples:

$890625^2 = \ldots 890625$, (*a*) $109376^2 = \ldots 109376$, (*b*) $890624^2 = \ldots 109376$, (*c*) $890624^3 = \ldots 890624$, $6249^3 = \ldots 6249$, (*d*) $3751^3 = \ldots 3751$.

Solutions (I.) *by* Lt.-Col. Allan Cunningham, R.E.;
(II.) *by* Rev. T. Wiggins, B.A., *and the* Proposer.

(I.) The definition of n involves

$$n^2 \equiv n \pmod{10^r} \quad \ldots\ldots(1).$$

Hence $$n \equiv n^2 \equiv n^3 \equiv \ldots \equiv n^k \pmod{10^r} \quad \ldots\ldots(2).$$

(*a*) Let $n' = 10^r+1-n$; therefore

$$n' \equiv -n+1 \pmod{10^r} \quad \ldots\ldots(3);$$

therefore $$n'^2 \equiv n^2-2n+1 \equiv -n+1 \equiv n' \pmod{10^r} \quad \ldots\ldots(4).$$

(*b*) $$(n-1)^2 = n^2-2n+1 \equiv -n+1 \equiv n' \quad \ldots\ldots(5).$$

(*c*) $$(n-1)^3 = n^3-3n^2+3n-1 \equiv n-1 \quad \ldots\ldots(6).$$

(*d*) Here the definition of n involves

$$n^3 \equiv n \pmod{10^r} \quad \ldots\ldots(7).$$

Let $10^r-n = n''$; therefore

$$n'' \equiv -n \pmod{10^r} \quad \ldots\ldots(8);$$

therefore $$n''^3 \equiv -n^3 \equiv -n \equiv n'' \quad \ldots\ldots(9).$$

Here the results (4), (5), (6), (9) prove the required theorems (*a*), (*b*), (*c*), (*d*).

(II.) Given $$(n^2-n) = \mathrm{M}(10^r) \quad \ldots\ldots(i.).$$

(*a*) To prove $[(10^r+1-n)^2-(10^r+1-n)] = \mathrm{M}(10^r)$.—Expression $= (10^r+1-n)(10^r-n) = \mathrm{M}(10^r)+(n^2-n) = \mathrm{M}(10^r)$, from (i.).

(*b*) To prove $(n-1)^2 = \mathrm{M}(10^r)+(10^r+1-n)$.—

$(n-1)^2 = (n^2-n)-(n-1) = \mathrm{M}(10^r)+(10^r+1-n)$, from (i.).

Digitized by Google

(*c*) To prove $(n-1)^3-(n-1) = \mathrm{M}(10^r)$.—

$(n-1)^3-(n-1) = (n-1)(n^2-n)-(n-1)(n) = \mathrm{M}(10^r)$, from (i.).

(*d*) To prove that, if $(n^3-n) = \mathrm{M}(10^r)$, so also shall $[(10^r-n)^3-(10^r-n)]$ be a multiple of 10^r.—

$[(10^r-n)^3-(10^r-n)] = \mathrm{M}(10^r)-n^3+n = \mathrm{M}(10^r)-(n^3-n) = \mathrm{M}(10^r)$.

15571. (Professor Neuberg.)—Soient Ox, Oy, Oz trois axes rectangulaires. On prend sur Ox, Oy les longueurs $OA = a$, $OB = b$. Soit C un point quelconque de la droite AB. Trouver l'équation de la surface engendrée par une ellipse qui a pour demi-axes OC, OD, la droite OD étant prise sur l'axe OZ et égale à K . OC, K étant un nombre donné.

Solution by Frances E. Cave.

Let C be (ξ, η). Then the equations of the ellipse are

$$x^2/\xi^2 + z^2/[\kappa^2(\xi^2+\eta^2)] = 1 \text{ and } x/\xi = y/\eta = \lambda, \text{ say.}$$

Also $\xi/a + \eta/b = 1$; therefore $\lambda = x/a + y/b$. Therefore the required equation is $(z^2/\kappa^2 + x^2 + y^2)(x/a + y/b)^2 = x^2 + y^2$.

15565. (Hon. G. R. Dick.)—If the bisectors of the base angles B and C of a triangle ABC meet the opposite sides in E and F, BC being given and the angle A constant, the line EF will touch a conic whose eccentricity is $\frac{1}{2}\operatorname{cosec}\frac{1}{2}A$.

Solution by Professor Sanjána, M.A.

Referring to BC and a perpendicular to it at its mid-point as axes, we have $\mathrm{BA} \equiv y = \tan \mathrm{B}\,(x+\frac{1}{2}a)$, $\mathrm{CF} \equiv y = -\tan\frac{1}{2}\mathrm{C}\,(x-\frac{1}{2}a)$;

so that at F $\quad y\cot \mathrm{B} - x - \frac{1}{2}a + \lambda\,(y\cot\frac{1}{2}\mathrm{C} + x - \frac{1}{2}a) = 0$.

Similarly, BE is $y\cot\frac{1}{2}\mathrm{B} - x - \frac{1}{2}a = 0$, and CA is $y\cot \mathrm{C} + x - \frac{1}{2}a = 0$;

hence at E $\quad y\cot\frac{1}{2}\mathrm{B} - x - \frac{1}{2}a + \mu\,(y\cot \mathrm{C} + x - \frac{1}{2}a) = 0$.

If we choose λ and μ so as to make the two equations identical, we get the equation of the line EF. Hence

$$\frac{\cot \mathrm{B} + \lambda\cot\frac{1}{2}\mathrm{C}}{\cot\frac{1}{2}\mathrm{B} + \mu\cot \mathrm{C}} = \frac{1-\lambda}{1-\mu} = \frac{1+\lambda}{1+\mu} = \frac{2}{2} = 1;$$

so that $\lambda = \mu$; hence $\lambda\,(\cot\frac{1}{2}\mathrm{C} - \cot \mathrm{C}) = \cot\frac{1}{2}\mathrm{B} - \cot \mathrm{B}$, *i.e.*,

$$\lambda = \sin \mathrm{C}/\sin \mathrm{B}.$$

Substituting and simplifying, we get

$$y\,(1 + \cos \mathrm{B} + \cos \mathrm{C}) - x\,(\sin \mathrm{B} - \sin \mathrm{C}) - \tfrac{1}{2}a\,(\sin \mathrm{B} + \sin \mathrm{C}) = 0,$$

Digitized by Google

Hence, putting $B-C = 2\theta$, we get

$$y(1+2\sin\tfrac{1}{2}A\cos\theta) - 2x\sin\tfrac{1}{2}A\sin\theta - a\cos\tfrac{1}{2}A\cos\theta = 0.$$

Eliminating θ between this and the differential coefficient equated to zero, we get for the envelope

$$(2y\sin\tfrac{1}{2}A - a\cos\tfrac{1}{2}A)^2 + 4x^2\sin^2\tfrac{1}{2}A = y^2;$$

whence $4x^2\sin^2\tfrac{1}{2}A + y(4\sin^2\tfrac{1}{2}A - 1) - 2ay\sin A + a^2\cos^2\tfrac{1}{2}A = 0.$

The equation for the eccentricity of this conic is seen to be

$$4\sin^2\tfrac{1}{2}A(4\sin^2\tfrac{1}{2}A - 1)e^4 + e^2 - 1 = 0;$$

so that $$(4e^2\sin^2\tfrac{1}{2}A - 1)[e^2(4\sin^2\tfrac{1}{2}A - 1) + 1] = 0.$$

If $4\sin^2\tfrac{1}{2}A > 1$, or $A > 60°$, there is only one real eccentricity, viz., $\frac{1}{2}\operatorname{cosec}\frac{1}{2}A$, less than unity, and the conic is an ellipse. If $4\sin^2\tfrac{1}{2}A < 1$, or $A < 60°$, both eccentricities are real, viz.,

$$\tfrac{1}{2}\operatorname{cosec}\tfrac{1}{2}A \quad \text{and} \quad 1/\sqrt{(1-4\sin^2\tfrac{1}{2}A)},$$

and greater than unity. Hence the envelope is a hyperbola with eccentricity $\frac{1}{2}\operatorname{cosec}\frac{1}{2}A$, the other value being the eccentricity of the conjugate hyperbola. When $A = 60°$ the envelope is a parabola.

15573. (Lt.-Col. Allan Cunningham, R.E.)—Factorise completely into *prime* factors $N = (208^{13}-1)$. [This contains 30 figures.]

Solution by the Proposer.

In general, let $N = (y^{13}-1) = N_{13}N_1$, where

$$N_{13} = (y^{13}-1)/(y-1), \quad N_1 = (y-1).$$

Now, when $y = 13\eta^2$, N_{13} is an Aurifeuillian of 13th order, whose factors are $N_{13} = L.M$, where $L = P-Q$, $M = P+Q$.

$$P = 1 + 7y + 15y^2 + 19y^3 + 15y^4 + 7y^5 + y^6,$$
$$Q = 13\eta(1 + 3y + 5y^2 + 5y^3 + 3y^4 + y^5)$$

(see Ed. Lucas's Memoir, *Sur la Série récurrente de Fermat*, Rome, 1879, where the formulæ are ascribed to M. Aurifeuille).

Ex.—Take $y = 208 = 13 \cdot 4^2$; then $\eta = 4$; $N_1 = 207 = 9 \cdot 23$;

$P = 83733967920305$; $\quad Q = 20539451990772$;

$L = P-Q = 63194515929533$; $\quad M = P+Q = 104273419911077.$

Now, the writer has compiled a table of solutions of the congruences $(y^{13}\mp 1)/(y\mp 1) \equiv 0 \pmod{p}$ for all primes (p) up to $p \not> 10{,}000$ (which it is hoped to publish in due course). This gives 53, 5591, 8971 as the only prime factors ($< 10{,}000$) of N_{13}. And, on actual trial division,

$$L = 53 \cdot 5591 \cdot 213262271; \quad M = 8971 \cdot 11623388687.$$

On testing the two large (9-figure and 11-figure) factors of L, M by Lawrence's process (see *Proc. Lond. Math. Soc.*, Vol. xxviii., p. 465), they are both found prime. Lastly, $N = N_1(L.M)$, and N is thus completely resolved.

Digitized by Google

15393. (Professor SANJÁNA, M.A.)—In the series

$$S \equiv u_0 + u_1 + u_2 + u_3 + u_4 + u_5 + \ldots,$$

$u_0 = 1$, $u_1 = p$; also $u_2 = au_1 - u_0$, $u_3 = bu_2 - u_1$, $u_4 = au_3 - u_2$, $u_5 = bu_4 - u_3$, ... Prove that when S is convergent its sum

$$= [2ap - ab - \sqrt{(a^2b^2 - 4ab)}]/2(ap^2 - apb + b).$$

Solution by H. J. WOODALL, A.R.C.S.

Let $S_e = u_0 + u_2 + u_4 + \ldots$, $S_o = u_1 + u_3 + u_5 + \ldots$; therefore $S = S_e + S_o$. The scale $u_2 - au_1 + u_0 = 0$ gives $2S_e - aS_o = u_0 = 1$. The scale $u_3 - bu_2 + u_1 = 0$ gives $2S_o - bS_e = u_1 - bu_0 = p - b$; whence

$$S_e(4 - ab) = 2 + ap - ab, \quad S_o(4 - ab) = b + 2p - 2b;$$

therefore

$$S(4 - ab) = 2 - ab - b + p(a + 2).$$

[Note the sum given in the question is one root of the quadratic equation

$$S^2(ap^2 - apb + b) - 2S(2ap - ab) + 4a = 0,$$

i.e., of

$$a(Sp - 2)^2 - abS(Sp - 2) + bS^2 = 0.]$$

14625 & 15546. (Professor NANSON.)—(14625.) Tangents are drawn from $x'y'z'$, $x''y''z''$ to conics touching the lines

$$x = 0, \quad y = 0, \quad z = 0, \quad lx + my + nz = 0;$$

show that the locus of their intersection is

$$(lx + my + nz)[l(x'x''/x) + m(y'y''/y) + n(z'z''/z)]$$
$$= (lx' + my' + nz')(lx'' + my'' + nz'').$$

(15546.) Find the locus of the foci of a conic touching four given lines, and deduce the locus of the focus of a parabola touching three given lines.

Solutions by the PROPOSER.

There are three points of the locus on P'P'', viz., P', P'', and the point of contact of P'P'' with the one conic of the system which it touches. Hence the required locus is a cubic through P', P'', and obviously passes through the six vertices, and through the meets of the joins of P', P'' with opposite vertices. If x, y, z, w are the four lines, it is at once verified that the cubic is

$$a(x'x''/x) + b(y'y''/y) + c(z'z''/z) + d('ww''/w) = 0,$$

where $ax + by + cz + dw = 0$ is the identical relation connecting x, y, z, w.

Replacing a, b, c, d by $l, m, n, -1$, we get 14625, and, taking P', P'' to be the focoids, we get 15546, since when one of the four lines coincides with P'P'' the cubic breaks up into the line P'P'' and a conic through P', P''.

15528. (A. M. NESBITT, M.A.)—A triangle ABC is circumscribed by a hyperbola whose asymptotes are parallel to the bisectors of the

Digitized by Google

angle A and its supplement. Find the angle at which the tangent at A cuts the opposite side.

Solution by C. M. Ross.

The asymptotes are perpendicular, since the bisectors of the angle A and supplement are perpendicular. The hyperbola is therefore equilateral.

Let the equation of the curve be $xy = c^2$. Let the co-ordinates of the angular points of the triangle be $(ct_1, c/t_1)$, $(ct_2, c/t_2)$, $(ct_3, c/t_3)$.

The equation of the bisectors of the angle A and supplement is

$$\frac{x + yt_1t_2 - c(t_1 + t_2)}{\sqrt{(1 + t_1^2t_2^2)}} = \pm \frac{x + yt_1t_3 - c(t_1 + t_3)}{\sqrt{(1 + t_1^2t_3^2)}},$$

i.e.,

$$\sqrt{(1 + t_1^2t_3^2)}\left\{x + yt_1t_2 - c(t_1 + t_2)\right\} = \pm \sqrt{(1 + t_1^2t_2^2)}\left\{x + yt_1t_3 - c(t_1 + t_3)\right\}.$$

Now, asymptotes are parallel to bisectors; hence

$$\sqrt{(1 + t_1^2t_2^2)} = \sqrt{(1 + t_1^2t_3^2)}, \quad i.e., \quad t_2 = \pm t_3.$$

[If $t_2 = t_3$, BC becomes a tangent.] Therefore, if $t_2 = -t_3$, the equation to BC becomes $x = yt_2^2$ .....(1).

The tangent at A is $x + yt_1^2 = 2ct_1$ ....(2).

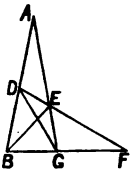
The angle between (1) and (2) is therefore

$$\tan^{-1} \frac{t_1^2 + t_2^2}{t_1^2t_2^2 - 1}.$$

15586. (*Communicated by* Frank Porter, M.A.)—ABC is an isosceles triangle, base angles being 80°;

$$\angle EBC = 50°, \quad \angle BCD = 60°;$$

DE is joined and produced to meet the base produced in F. Find the angle EFC by means of propositions in Euclid, Book I. (This problem resembles Question 15474, except that it is capable of solution by elementary geometry.)



Solution by the Proposer *and others.*

Draw DH parallel to BC, meeting AC in H. Join DC, BH, intersecting in G. Join GE. The triangles GBC, GHD are equilateral. Also the triangle EBC is isosceles. Since $\angle EBC = 50°$, $\angle BCE = 80°$; therefore $\angle BEC = 50°$; therefore CE = BC = CG; therefore

$$\angle CGE = \angle CEG = 80°;$$

therefore $\angle DGE = 100°$ and $\angle DGH = 60°$; therefore $\angle HGE = 40°$; therefore $\angle GHE = \angle GEC - \angle HGE = 40°$; therefore GE = HE and DG = DH; therefore $\angle HDE = \angle GDE = \frac{1}{2}$ of $60° = 30° = EFC$.

Digitized by Google

15516. (Hon. G. R. Dick.)—The surface whose co-ordinates at any point are given by

$$x = \rho\cos\phi,\quad y = \rho\sin\phi,\quad z = b\phi + c\int[(\rho^2+b^2)/(\rho^2-c^2)]^{\frac{1}{2}}\,d\rho/\rho$$

is a minimum surface: exhibit them in Monge's form; also the surface $\sin z = \sinh x \sinh y$.

Solution by the Proposer.

(1) It will be found that the values

$$4x = (c+bi)(a+1/a)+(c-bi)(\beta+1/\beta),$$
$$4yi = (c+bi)(a-1/a)-(c-bi)(\beta-1/\beta),$$
$$2z = -(c+bi)\log a+(c-bi)\log\beta$$

satisfy the equation, a and β being arbitrary.

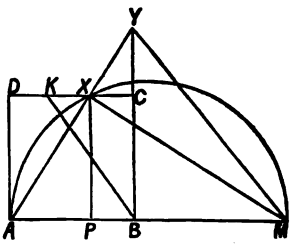
(2) With regard to the surface $\sin z = \sinh x \sinh y$, the forms are

$$x = \tfrac{1}{2}\log(1+ai)/(1-ai)+\tfrac{1}{2}\log(1-\beta i)/(1+\beta i),$$
$$y = \tfrac{1}{2}\log(1+a)/(1-a)+\tfrac{1}{2}\log(1+\beta)/(1-\beta),$$
$$iz = \tfrac{1}{2}\log(1-a^2)/(1+a^2)+\tfrac{1}{2}\log(1+\beta^2)/(1-\beta^2).$$

15102. (J. Prescott, B.A.)—Given a square ABCD; draw a line through A to meet CD in X and BC produced in Y so that XY may be equal to a given length.

Solutions (I.) *by the* Proposer; (II.) *by* C. E. Youngman, M.A.; (III.) *by* D. Biddle.

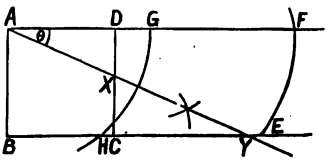
(I.) Let CK be the given length to which XY must be equal. On AB produced cut off BM = BK. On AM describe the semicircle AXM, meeting DC in X. Join AX and produce to meet BC in Y. Then XY = CK. Draw XP parallel to CB. Then the triangles XPM and ABY are similar and equal.



Now $XY^2+XM^2 = MY^2$
$= BY^2+BM^2$
$= BY^2+BK^2 = BY^2+BC^2+CK^2 = PM^2+BC^2+CK$
$= XM^2+CK^2$;

therefore $XY = CK$.

(II.) Writing θ for the angle DAX, a for the side of the square, and d for the given length, we have $a = a\tan\theta + d\sin\theta$
or $d/a = (\cos\theta-\sin\theta)/\sin\theta\cos\theta$;
$\therefore\ d^2/4a^2 = (1-\sin 2\theta)/\sin^2 2\theta$;
$\therefore\ 2a\operatorname{cosec} 2\theta = a+\sqrt{(a^2+d^2)}$.



Digitized by Google

Hence the construction :—On BC produced take CE, the given length; on AD produced DF = DE; bisect AF at G; with centre A and radius AG cut BC at H; and draw AXY bisecting the angle GAH.

(III.) Let AB = unity,

$XY = 2a$, $\angle BAY = \theta$.

Then $\sec\theta - \operatorname{cosec}\theta = 2a$,

whence

$$\sin\theta - \cos\theta = a \sin 2\theta,$$

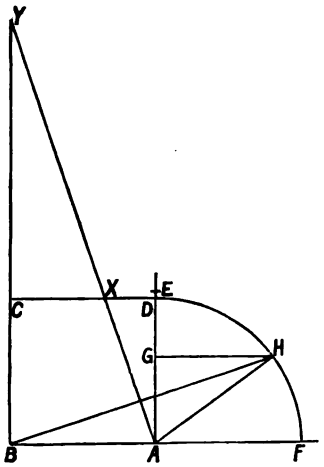
or $[\frac{1}{2}(1-\cos 2\theta)]^{\frac{1}{2}} - [\frac{1}{2}(1+\cos 2\theta)]^{\frac{1}{2}} = a \sin 2\theta.$

Squaring both sides, we have

$$1 - \sin 2\theta = a^2 \sin^2 2\theta,$$

whence

$$\sin 2\theta = 1/a[(1 + 1/4a^2)^{\frac{1}{2}} - 1/2a].$$



This value is easily obtainable geometrically, but to represent the process would complicate the diagram. Therefore, let AG = the value of $\sin 2\theta$ in terms of a, as given above, the value of a being represented by AE. With radius AD describe the arc DF, to meet BA (produced) in F. Draw GH parallel to BA to meet the arc in H, and join BH. It is easy to see that the required line AY intersects BH at right angles.

15600. [*Communicated by* Rev. T. Roach, M.A.]—Four pennies are placed flat on a table so that each touches two of the others. Find when the space enclosed between them is a minimum or maximum.

Solution by Rev. T. Wiggins, B.A.

Let r be the radius of a penny and let α be an angle of the rhombus formed by joining the centres of the touching circles.

Required area = area of rhombus − sum of sectors $= 4r^2 \sin\alpha - \pi r^2$.

Hence the area is a maximum when $\alpha = 90°$. The minimum value of α is 60°, viz., when the opposite pennies touch; therefore

$$\text{maximum} = r^2(4-\pi), \quad \text{minimum} = r^2(2\sqrt{3}-\pi).$$

Note.—The Editor remarks that strictly speaking 60° is the lower *limit* of α and is never actually reached unless the words of the Question, "Each touches two of the others," are interpreted "Each touches *at least* two of the others": for when $\alpha = 60°$ in two cases out of the four a penny touches each of the other three.

Digitized by Google

15479. (H. G. Graves.)—If x be the area of one of the faces of a tetrahedron; A, B, C the dihedral angles over x; and

$$M = \sqrt{(1-\cos^2 A-\cos^2 B-\cos^2 C-2\cos A\cos B\cos C)},$$

show that A/M has the same value for all the solid angles.

Solution by Professor Nanson.

Project on the face f_q the other three faces; then we get $\Sigma_p (pq) f_p = 0$, where (pq) is the cosine of the angle between the faces f_p, f_q and $(pp) = -1$. Hence the determinant $|(pq)|$ vanishes and the co-factors of parallel lines therein are therefore proportional, so that, if $[pq]$ is the co-factor of (pq), we have $f_1^2, f_2^2, \ldots; f_1, f_2, \ldots$ proportional to $[11], [22], \ldots; [12], \ldots$, which gives the required result.

15549. (A. M. Nesbitt, M.A.)—CP, CQ are conjugate semi-diameters of an ellipse. If the triangle CPQ be of constant area, prove that the envelope of PQ is a similar ellipse.

Solutions (I.) *by* Professor Sanjána, M.A.; (II.) *by* W. Scrimgeour, M.A., B.Sc., *and others.*

(I.) If CP, CQ are conjugate, the area CPQ is always constant and equal to the area CAB, where CA, CB are principal semi-axes. But the result is true even when CP, CQ are not conjugate. Let the eccentric angles of P, Q be α, β respectively; then $\Delta CPQ = \frac{1}{2}ab\sin(\alpha-\beta)$; hence for area to be constant $\alpha-\beta$ must be fixed and equal to 2ϕ, suppose. The equation to the chord PQ is then seen to be

$$x\cos\tfrac{1}{2}(\alpha+\beta)/a + y\sin\tfrac{1}{2}(\alpha+\beta)/b = \cos\phi;$$

and by ordinary working we get for the envelope $x^2/a^2 + y^2/b^2 = \cos^2\phi$, which is a similar and concentric ellipse. In the case when CP, CQ are conjugate, $2\phi = \frac{1}{2}\pi$, and the envelope becomes $x^2/\frac{1}{2}a^2 + y^2/\frac{1}{2}b^2 = 1$.

(II.) The triangle CPQ is always constant.

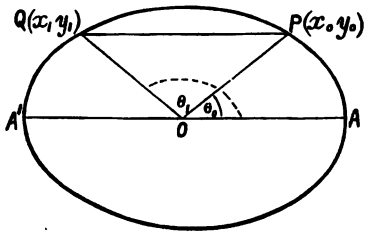
$P = (x_0, y_0)$, $Q = (x_1, y_1)$.

Let $\angle ACP = \theta_0$,

$\angle ACQ = \theta_1$,

$CP = l_0$, $CQ = l_1$;

also let the eccentric angle of P be ϕ; then the co-ordinates of P are $a\cos\phi$, $b\sin\phi$; then the eccentric angle of Q is $\phi + \frac{1}{2}\pi$, the co-ordinates of Q are $-a\sin\phi$, $b\cos\phi$.



$$2\Delta PCQ = l_0 l_1 \sin(\theta_1-\theta_0) = l_0\cos\theta_0\, l_1\sin\theta_1 - l_0\sin\theta_0\, l_1\cos\theta_1$$
$$= x_0y_1 - x_1y_0 = a\cos\phi\, b\sin(\phi+\tfrac{1}{2}\pi) - a\cos(\phi+\tfrac{1}{2}\pi)\, b\sin\phi$$
$$= ab\sin\tfrac{1}{2}\pi = ab.$$

Digitized by Google

The equation of the line PQ is

$$x/a \cos \tfrac{1}{2}[\phi + (\phi + \tfrac{1}{2}\pi)] + y/b \sin \tfrac{1}{2}[\phi + (\phi + \tfrac{1}{2}\pi)] = \cos \tfrac{1}{2}[\phi - (\phi + \tfrac{1}{2}\pi)]$$

or $$x/a \cos(\tfrac{1}{4}\pi + \phi) + y/b \sin(\tfrac{1}{4}\pi + \phi) = 1/\sqrt{2},$$

i.e., $$x/a(\cos\phi - \sin\phi) + y/b(\cos\phi + \sin\phi) = 1.$$

Now, if l and m in the equation $lx + my = 1$ satisfy the equation $a^2l^2 + b^2m^2 = 1$, the line will always touch the ellipse $x^2/a^2 + y^2/b^2 = 1$.

Here $$[a^2(\cos\phi - \sin\phi)^2]/a^2 + [b^2(\cos\phi + \sin\phi)^2]/b^2 = 2$$

or $$(a/\sqrt{2})^2[(\cos\phi - \sin\phi)^2]/a^2 + (b/\sqrt{2})^2[(\cos\phi + \sin\phi)^2]/b^2 = 1;$$

therefore PD always touches the ellipse $x^2/(a/\sqrt{2})^2 + y^2/(b/\sqrt{2})^2 = 1$. $2b^2x^2 + 2a^2y^2 = a^2b^2$, a similar ellipse.

15580. (Professor SANJÂNA, M.A.)—A parabola has contact of the second order with $y^2 = 4ax$ at the vertex, and cuts it again at the point whose normal is inclined at $\tan^{-1}m$ to the axis. Prove that its equation is $(4x + my)^2 = 4am^2x$.

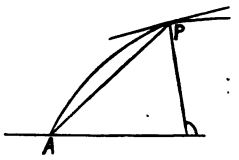
Solutions (I.) *by* R. TUCKER, M.A.; F. W. REEVES, B.A., *and others*; (II.) *by* FRANCES E. CAVE, C. A. B. GARRETT, *and* A. M. NESBITT, M.A.

(I.) The equation to AP is

$$my + 2x = 0;$$

hence the curve is

$$y^2 - 4ax - \lambda x(my + 2x) = 0 \quad \ldots\ldots(\text{i.}).$$



Hence, for (i.) to be a parabola, we must have $4 \cdot 1(-2\lambda) = \lambda^2m^2$, *i.e.*, $m^2\lambda = -8$, and (i.) becomes $(4x + my)^2 = 4am^2x$ (*cf.* C. Smith, *Conics*, § 230).

(II.) Let the equation of the parabola be $(\alpha x + \beta y)^2 + px + qy + r = 0$. Then 0, 0, 0, $-2am$ are roots of $(\alpha y^2/4a + \beta y)^2 + p(y^2/4a) + qy + r$; therefore $r = 0$, $q = 0$, $4a\beta^2 + p = 0$, $2\beta = 2am(a/4a)$; therefore the equation may be written $(4x + my)^2 = 4am^2x$.

15521. (Lt.-Col. ALLAN CUNNINGHAM, R.E.)—If σN denote the sum of the sub-factors of N (including 1, but excluding N), obtain some solutions of $\sigma N = X^2$ (an *even* square).

Solutions (I.) *by the* PROPOSER; (II.) *by* JAMES BLAIKIE, M.A.

(I.) Solutions of this question may be obtained in many ways. The process being indirect, the form of solution must be assumed. One such process is: Let $P = 2^{q-1}M_q$, where $M_q = 2^q - 1$, a prime (q being prime). Also let $N = P.p$, where p is an odd prime, but $\neq M_q$. Let $\int N$ denote

Digitized by Google

the sum of all the divisors of N. Then, P being a perfect number, $\int P = 2P$, also $\int p = p+1$. And, by definition,

$$\sigma N = \int Pp - Pp = \int P . \int p - Pp ;$$

therefore

$$X^2 = \sigma N = 2P(p+1) - Pp = P(p+2) = 2^{q-1} . M_q (p+2).$$

This involves $p = M_q - 2 = 2^q - 3$, a prime. Then there is a solution whenever (2^q-1), (2^q-3) are both prime, which gives

$$N = 2^{q-1}(2^q-1)(2^q-3), \quad X = 2^{\frac{1}{2}(q-1)}(2^q-1).$$

Ex.—$q = 3$ gives $N = 2^2 . 7 . 5$, $X = 2 . 7$; $q = 5$ gives $N = 2^4 . 31 . 29$, $X = 2^2 . 31$, and there are no more of this form up to $q = 149$ at least. Another form giving many cases is $N = 2^r . p$.

(II.) Let $X = 2m$. If $4m^2 - 3 = p$, a prime, then, if $N = 2p$, its factors are $\sigma N = 1 + 2 + p = 3 + p = 4m^2 = X^2$.

Thus N satisfies the condition. *E.g.*, let $m = 2$, $p = 13$, $N = 26$. Similarly we obtain 122, 194, 386, 794, 1146, ... Again, if $4m^2 - 7 = 3p$, where p is prime, then $N = 4p$ satisfies the condition. *E.g.*, $m = 2$, $p = 3$, $N = 12$. Similarly we obtain 76, 124, 332, ... Other solutions are $N = 8p$, where $4m^2 - 15 = 7p$, giving 24, 56, ..., and $N = 18p$, where $4m^2 - 39 = 21p$, giving 90, ... And so on.

Remarks on Mr. A. M. Nesbitt's *Note on the Solution of certain Equations.*

By Professor Sanjána, M.A.

[See *supra*, pp. 41 and 42.]

It is undoubtedly true that the equality

$$\frac{x}{a+\theta} + \frac{y}{b+\theta} + \frac{z}{c+\theta} - 1 = \frac{(\lambda-\theta)(\mu-\theta)(\nu-\theta)}{(a+\theta)(b+\theta)(c+\theta)} \quad \ldots\ldots\ldots\ldots(\alpha)$$

is satisfied by $\theta = 0$; this is due to its being an identity, and other values of θ, though not so simple, must also satisfy it. This does not vitiate the process, which consists not in solving for θ from a cubic or quadratic, but in equating the coefficients of the various powers in $A + B\theta + C\theta^2 = 0$ to zero. This gives three equations in x, y, z, one of which is

$$\Sigma bcx = abc + \lambda\mu\nu ;$$

the other two are also simple equations, so that x, y, z are determined uniquely. The final result is the same, but it is best to obtain it in the ordinary fashion from (α), by multiplying successively by $a+\theta$, $b+\theta$, $c+\theta$, and then putting $\theta = -a$, $-b$, $-c$, respectively, as some difficult factorization is thereby avoided. The identity is certainly satisfied by making θ infinite.

15512. (R. Tucker, M.A.)—PQ is a chord of a parabola, constant in direction. PR, QR are also chords drawn in a constant direction. Show that the locus of R is a parabola.

Digitized by Google

Solution by Professor NANSON.

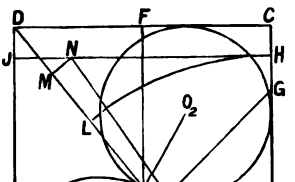
Consider the more general problem of finding the locus of the vertex R of a triangle whose base angles P, Q lie on a fixed conic S, and whose sides QR, RP, PQ pass through fixed points A, B, C. Clearly there are four positions of R on AB, viz., two at A and two at B. The locus of R is therefore a quartic with nodes at A, B, and passes through the six meets of S with CA, CB, and the polar of C. If now A, B, C are collinear, the quartic breaks up into a conic S′ and the line ABC counted twice. The meets of S′ with ABC are clearly real or imaginary according as the meets of S with ABC are real or imaginary. Hence, if ABC touches S, it also touches S′. Thus in the proposed Question the required locus is a parabola whose axis is parallel to that of the given parabola.

In the general case the remaining two meets of S and the quartic give the solution of Castillon's problem to inscribe in S a triangle whose sides pass through fixed points. (*Cf.* Salmon, Art. 297, Ex. 7.) When S is a pair of lines the quartic breaks up into two MacLaurin conics.

15567. (SOLIDUS.)—Two equal circular discs are to be cut out of a rectangular piece of paper 10 inches long and 8 inches broad. What is the greatest possible diameter of the discs?

Solutions (I.) *by* D. BIDDLE; (II.) *by* JAMES BLAIKIE, M.A., *and* B. C. WALLIS, F.C.P.; (III.) *by* GEORGE SCOTT, M.A., *and* I. ARNOLD.

(I.) It is easy to see that each circle touches two sides, and that the two circles touch at the centroid of the rectangle.



Construction. — Make BG = BA; bisect GC in H, and draw HJ parallel to CD. With centre B, draw the arc HL to meet the diagonal BD in L, and make LM = GC. With centre B, draw the arc MN to meet HJ in N, and make HP = HN. Draw PQ parallel to AB, and produce to meet AG in O_1; join O_1K, and produce, making $KO_2 = O_1K$. Then O_1, O_2 are the centres of the required circles. For

$$BG = BA = 8, \quad BL = BH = 9, \quad BN = BM = 11,$$

$$HP = HN = \sqrt{(11^2-9^2)}, \text{ and } QE = PB = 9-\sqrt{(11^2-9^2)}.$$

But $$QE = 5-KQ = 4-SQ = 4-QO_1.$$

Moreover, $$KO_1^2 = (1+QO_1)^2+QO_1^2 = QE^2 = (4-QO_1)^2,$$

Digitized by Google

whence $QO_1^2 + 10QO_1 = 15$. Therefore

$$QO_1 = 2\sqrt{(10)} - 5 = \sqrt{(11^2 - 9^2)} - 5,$$

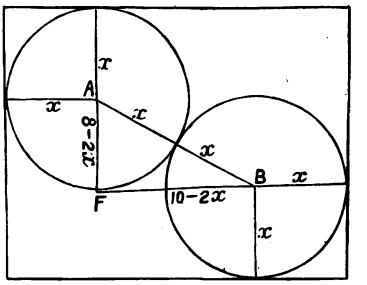
which gives $QE = 4 - QO_1 = 9 - \sqrt{(11^2 - 9^2)}$,

agreeing with the above. Hence, the required diameter

$$= 18 - 2\sqrt{(11^2 - 9^2)} = 18 - 4\sqrt{(10)}.$$

(II.) Let A, B be the centres of the discs and let x be their common radius. Then the sides of the right-angled triangle AFB (see figure) are

$$AB = 2x, \quad AF = 8 - 2x,$$
$$BF = 10 - 2x;$$



therefore

$$4x^2 = 4(4-x)^2 + 4(5-x)^2$$

or $x^2 - 18x + 41 = 0$;

therefore

$$x = 9 - \sqrt{(40)} = 2{\cdot}676$$

approximately; therefore greatest diameter = 5·35 inches (nearly).

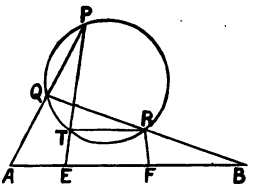
(III.) In order that the two equal circles may be the largest possible, each must touch a pair of the rectangle's adjacent sides; they must also touch one another. They can therefore be found, as proved in Question 15260 (October, 1903), thus—describe a circle passing through the intersection of the diagonals and touching two adjacent sides: this will be one of the required circles. Its radius can be found by the quadratic $(4-r)^2 + (5-r)^2 = r^2$, the half sides of the rectangle being 4 and 5 inches; whence $2r = 18 - 4\sqrt{(10)} = 5{\cdot}3508$ nearly. This common diameter of the two required circles is plainly greater than that of the circles inscribed in either half rectangles; for it is $18 - 2\sqrt{(40)}$, while the diameter of the in-circles is $18 - 2\sqrt{(41)}$.

Geometrical Note on Inversion.

By R. F. Davis, M.A.

(*Extracted by kind permission from the "Mathematical Gazette."*)

Let A, B be fixed points, P a variable point; $AQ \cdot AP = k^2$ and $BR \cdot BQ = k^2$; so that R is the inverse of the inverse of P.



About PQR describe a circle and draw the chord RT parallel to AB. Produce PT to meet AB in E. Then

$$\angle PEB = \angle PTR = \angle PQR \ (\angle PQB);$$

therefore P, Q, E, B are concyclic and $AE \cdot AB = AQ \cdot AP = k^2$. Therefore E is a fixed point. From R draw

Digitized by Google

RF, making $\angle RFB = \angle TEA$, so that TEFR is a symmetrical trapezium and FR = ET. Then $\angle RFB = \angle TEA = \angle ETR = \angle AQR$; therefore Q, A, F, R are concyclic and $BF \cdot BA = BR \cdot BQ = k'^2$. Therefore F is a fixed point.

Again, from the similar triangles PAE, BRF, EA : EP = FR : FB; wherefore FR . EP (or ET . EP) = EA . FB = constant.

Hence, if P describe any curve, R describes the geometrical image in the perpendicular bisector of EF of the inverse of the curve with respect to E. Thus, no matter what centres and constants of inversion are taken, repeated inversions of any plane curve will only give different inversions of a series of reflections of the original curve.

15556. (Lt.-Col. ALLAN CUNNINGHAM, R.E.)—Factorise completely into *prime* factors $N = G_4^4 + 8 + F_4^4$, where $G_n = F_0 . F_1 . F_2 . \ldots . F_{n-1}$ and $F_n = 2^{2^n} + 1$, a Fermat's number.

Solution by the PROPOSER.

Let $E_n = 2^{2^n}$; then $E_{n+1} = E_n^2$ and $F_n = E_n + 1$.

$$G_4 = (2^1 - 1)[(2^1 + 1)(2^2 + 1)(2^4 + 1)(2^8 + 1) = (2^{16} - 1) = E_4 - 1.$$

Similarly $G_n = E_n - 1$, $N_n = G_n^4 + 8 + F_n^4 = (G_n^4 + 4 . 1^4) + (F_n^4 + 4 . 1^4)$. Each of these terms is a Bin-Aurifeuillian; therefore

$$\begin{aligned} N_n &= [(G_n - 1)^2 + 1^2][(G_n + 1)^2 + 1^2] + [(F_n - 1)^2 + 1^2][(F_n + 1)^2 + 1^2] \\ &= [(E_n - 2)^2 + 1](E_n^2 + 1) + (E_n^2 + 1)[(E_n + 2)^2 + 1] \\ &= (E_{n+1} + 1)\, 2\,(E_n^2 + 5) = 2\,(E_{n+1} + 1)(E_{n+1} + 5). \end{aligned}$$

Ex.—Taking $n = 4$,

$$\begin{aligned} N &= G_4^4 + 8 + F_4^4 = 2\,(2^{32} + 1)(2^{32} + 5) \\ &= 2\,(641 . 6700417)(27 . 47 . 3384529). \end{aligned}$$

10582. (R. KNOWLES, B.A.)—The equation to a conic referred to a tangent and normal as axes being $ax^2 + bxy + cy^2 + gy = 0$, prove that, e being the eccentricity, the equation to its transverse axis is

$$\{4a(a+c) - (2-e^2)(b^2 - 4ac)\}x + 2b(a+c)y + bg(2-e^2) = 0.$$

[N.B.—For $(b^2 - 4ac)$ read $(4ac - b^2)$.—ED.]

Solution by FRANCES E. CAVE *and* L. ISSERLIS, B.A.

Let the equation referred to the principal axes be

$$\xi^2/a^2 + \eta^2/[a^2(1-e^2)] = 1,$$

and let (ξ_1, η_1) be foot of normal. Then

$$\xi = \xi_1 + \frac{p}{a^2}\left(\frac{x\eta_1}{1-e^2} + y\xi_1\right), \quad \eta = \eta_1 + \frac{p}{a^2}\left(\frac{y\eta_1}{1-e^2} - x\xi_1\right);$$

Digitized by Google

therefore

$$(1-e^2)\left[\frac{p}{a^2}\left(\frac{x\eta_1}{1-e^2}+y\xi_1\right)+\xi_1\right]^2+\left[\frac{p}{a^2}\left(\frac{y\eta_1}{1-e^2}-x\xi_1\right)+\eta_1\right]^2-a^2(1-e^2)$$
$$\equiv \lambda(ax^2+bxy+cy^2+gy).$$

Equating coefficients, $\lambda(a+c)=2-e^2$, $\lambda^2(b^2-4ac)=4(e^2-1)$,

$$\lambda a = 1-\frac{e^2p^2\eta_1^2}{a^4(1-e^2)^2}, \quad \lambda b = -\frac{2e^2p^2\eta_1\xi_1}{a^4(1-e^2)}, \quad \lambda g = \frac{2a^2(1-e^2)}{p};$$

therefore
$$\frac{\xi_1^2p^2}{a^4}=\frac{\lambda a+e^2-1}{e^2}.$$

The equation of the transverse axis is

$$0=\eta=\frac{1}{\xi_1}\left(\xi_1\eta_1+\frac{p}{a^2}\frac{y\xi_1\eta_1}{1-e^2}-px\frac{\xi_1^2}{a^2}\right)$$

or
$$x[4a(a+c)+(2-e^2)(b^2-4ac)]+2b(a+c)y+bg(2-e^2)=0.$$

15581. (C. E. YOUNGMAN, M.A.)—If A, B, C, D, E be five points on a rectangular hyperbola, A, B, C, D concyclic, DD′ a diameter of the circle, EE′ of the hyperbola, then the conics ABCD′E and ABCD′E′ will be ellipse and hyperbola, with their axes parallel and proportional.

Solution by W. F. BEARD, M.A.

Let CD′, CE, CE′ meet AB at K, L, M respectively. Let p, d be the diameters of the conic ABCD′E parallel to AB, CE; p_1, d_1 those of ABCD′E′ parallel to AB, CE′; P, D, D_1 those of the rectangular hyperbola ABCD parallel to AB, CE, CE′. Since A, B, C, D are on a circle, therefore the diameters of the rectangular hyperbola parallel to AB, CD are equal. But CD′ is perpendicular to CD; therefore AB, CD′ are parallel to conjugate diameters of the rectangular hyperbola; therefore the bisectors of the angle AKC are parallel to the asymptotes of the rectangular hyperbola.

Now, since A, B, D′, C are on a circle, therefore

the bisectors of the angle AKC are parallel to the axes of the conics ABCD′E, ABCD′E′ .....(1).

Again, $MA.MB : ME.MC = p_1^2 : d_1^2 = P^2 : D_1^2$;

therefore $p_1 : d_1 = P : D_1$. Similarly, $p : d = P : D$.

Now CE, CE′, being supplemental chords of the rectangular hyperbola, are parallel conjugate diameters; therefore $D = D_1$; therefore

$$p : d = p_1 : d_1 \quad \text{.....(2).}$$

Also CE, CE′ are equally inclined to either asymptote of the rectangular hyperbola, and therefore to either axis of the conics ABCD′E, ABCD′E′; hence the diameter of the conic ABCD′E′ parallel to CE $= d_1$. Hence, from (1) and (2), the conics ABCD′E, ABCD′E′ have their axes parallel and the diameters in two separate directions are proportionals; therefore their axes are proportionals.

By considering which of the points K, L, M lie in AB, CD′ or in these lines produced, it will be plain that one of the conics is an ellipse and the other a hyperbola.

Digitized by Google

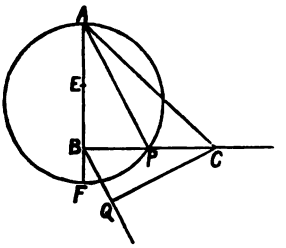
15400. (D. Biddle.)—AB, BC are two given straight lines which meet at right angles in B. Find in BC a point P such that, if BQ be drawn parallel to AP and CQ be perpendicular to it, CQ = 2BP. [A solution by plane geometry is required.]

Solutions (I.) *by* R. F. Davis, M.A., *and* Professor Joseph Brown; (II.) *by* James Blaikie, M.A.; (III.) *by* R. Tucker, M.A., *and others.*

(I.) Bisect AB in E and produce AEB to F, so that EF = ½AC. Upon AF as diameter describe a circle cutting BC in P, which will be the point required. For

$$PF^2 = AF \cdot BF = EF^2 - EA^2$$
$$= \tfrac{1}{4}(AC^2 - AB^2) = \tfrac{1}{4}BC^2.$$

Also CQ : BC = BP : PF;
and, since BC = 2PF,
therefore CQ = 2BP.

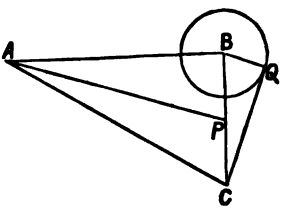


(II.) With centre B and radius equal to the difference between AC and AB describe a circle and draw CQ a tangent to the circle. Then, if AP be parallel to BQ, CQ = 2PB.

Proof.—

$$(AB + BQ)^2 = AB^2 + 2AB \cdot BQ + BQ^2$$
$$= AC^2 = AB^2 + BC^2;$$
$$\therefore \quad 2AB \cdot BQ = BC^2 - BQ^2 = CQ^2;$$
$$\therefore \quad CQ/AB = 2(BQ/CQ) = 2(PB/AB),$$

by similar triangles CBQ, APB; therefore CQ = 2PB.



(III.) Bisect AB in D and mark off AE = AC. Upon DE describe a semi-circle DPE, cutting BC in P; this is the point required. For, from the similar triangles CQB, ABP, $CQ/CB = AB/[\sqrt{(AB^2 + BP^2)}]$. Now CQ = 2BP, if $4BP^4 + 4AB^2 \cdot BP^2 = AB^2 \cdot CB^2$, that is, if $2BP^2 + AB^2 = AB \cdot CA$, that is, if $2BP^2 = AB(CA - AB) = AB \cdot BE$, that is, if $BP^2 = DB \cdot BE$; and this is so.

15595. (Professor Sanjána, M.A.)—Prove that

$$\cosh \tfrac{1}{2}u \cosh \tfrac{1}{4}u \cosh \tfrac{1}{8}u \ldots \text{ad inf.} = (\sinh u)/u.$$

Hence deduce the sums of the series, ad inf.,

(1) $\dfrac{x}{1+x} + \tfrac{1}{2}\dfrac{x^2}{1+x^2} + \tfrac{1}{4}\dfrac{x^4}{1+x^4} + \ldots$, (2) $\dfrac{1}{1+x} + \tfrac{1}{2}\dfrac{1}{1+x^2} + \tfrac{1}{4}\dfrac{1}{1+x^4} + \ldots$.

[The first result corresponds to Euler's formula for the circular functions; in the two series x may be less or greater than unity.]

Digitized by Google

Solutions (I.) *by* Rev. T. WIGGINS, B.A.; (II.) *by* C. M. ROSS *and* H. C. KENT.

(I.) $\sinh u = \cosh \frac{1}{2}u \, 2 \sinh \frac{1}{2}u = \cosh \frac{1}{2}u \cosh \frac{1}{4}u \, 2^2 \sinh (u/2^2)$

$= \cosh \frac{1}{2}u \cosh \frac{1}{4}u \ldots \cosh (u/2^n) \, [2^n \sinh (u/2^n)]$.

And $2^n \sinh (u/2^n) \, (n = \infty) = 2^n (u/2^n) = u$;

therefore $(\sinh u)/u = \cosh \frac{1}{2}u \cosh \frac{1}{4}u \cosh \frac{1}{8}u \ldots$.

Now call the required sums S and S'. From above we have

$\log (\sinh u)/u = \log \cosh \frac{1}{2}u \cosh \frac{1}{4}u \cosh \frac{1}{8}u \ldots$;

therefore

$\log (\sinh u)/u$

$= \log [(\epsilon^u + 1)/2\epsilon^{\frac{1}{2}u}] [(\epsilon^{\frac{1}{2}u} + 1)/2\epsilon^{\frac{1}{4}u}] [(\epsilon^{\frac{1}{4}u} + 1)/2\epsilon^{\frac{1}{8}u}] \ldots$,

$\log (\epsilon^u - \epsilon^{-u}) - \log 2u$

$= \log (1 + \epsilon^u) + \log (1 + \epsilon^{\frac{1}{2}u}) + \log (1 + \epsilon^{\frac{1}{4}u}) + \ldots - \log 2\epsilon^{\frac{1}{2}u} \cdot 2\epsilon^{\frac{1}{4}u} \ldots$.

Differentiatiate both sides; therefore

$(\epsilon^u + \epsilon^{-u})/(\epsilon^u - \epsilon^{-u}) - 1/u$

$= \epsilon^u/(1 + \epsilon^u) + \frac{1}{2} [\epsilon^{\frac{1}{2}u}/(1 + \epsilon^{\frac{1}{2}u})] + \ldots - (\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots)$.

Put $x = \epsilon^u$; therefore $(x + x^{-1})/(x - x^{-1}) - 1/\log x = S - 1$; therefore

$S = 2x^2/(x^2 - 1) - 1/\log x$.

Also $S' + S = 1 + \frac{1}{2} + \frac{1}{4} + \ldots = 2$; therefore

$S' = 2 - S = 1/\log x - 2/(x^2 - 1)$.

(II.) In Euler's formula, viz.,

$\cos (\theta/2) \cos (\theta/2^2) \cos (\theta/2^3) \cos (\theta/2^4) \ldots = (\sin \theta)/\theta$,

put $\theta = iu$; then

$\cos (iu/2) \cos (iu/2^2) \cos (iu/2^3) \cos (iu/2^4) \ldots = (\sin iu)/iu$,

therefore $\cosh (u/2) \cosh (u/2^2) \cosh (u/2^3) \cosh (u/2^4) \ldots = (\sinh u)/u \ldots (1)$.

Again, taking logs of both sides of (1),

$\log \cosh \frac{1}{2}u + \log \cosh \frac{1}{4}u + \log \cosh \frac{1}{8}u + \ldots = \log \sinh u - \log u$.

Differentiating, the result is

$\frac{1}{2} \tanh \frac{1}{2}u + \frac{1}{4} \tanh \frac{1}{4}u + \frac{1}{8} \tanh \frac{1}{8}u + \ldots = \coth u - 1/u \quad \ldots\ldots (2)$.

This identity may be written

$\frac{1}{2} (1 + \tanh \frac{1}{2}u) + \frac{1}{4} (1 + \tanh \frac{1}{4}u) + \frac{1}{8} (1 + \tanh \frac{1}{8}u) + \ldots$

$= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} \ldots \infty + \coth u - 1/u = 1 + \coth u - 1/u \ldots (3)$.

Now put $x = \epsilon^u$, so that (3) becomes

$$\frac{x}{1+x} + \frac{1}{2}\frac{x^2}{1+x^2} + \frac{1}{4}\frac{x^4}{1+x^4} + \ldots = 1 + \frac{x^2+1}{x^2-1} - \frac{1}{\log_e x} = \frac{2x^2}{x^2-1} - \frac{1}{\log_e x}.$$

Again, (2) may be written

$\frac{1}{2} (1 - \tanh \frac{1}{2}u) + \frac{1}{4} (1 - \tanh \frac{1}{4}u) + \frac{1}{8} (1 - \tanh \frac{1}{8}u) + \ldots = 1 + 1/u - \coth u$,

and, with the same substitution as before, the identity becomes

$$\frac{1}{1+x} + \frac{1}{2}\frac{1}{1+x^2} + \frac{1}{4}\frac{1}{1+x^4} + \ldots = \frac{1}{\log_e x} - \frac{2}{x^2-1}.$$

11375. (Professor ZERR.)—Find (i.) the maximum rectangle, (ii.) the maximum parallelopiped, inscribed in, first, the curve $(x/a)^{\frac{2}{3}} + (y/b)^{\frac{2}{3}} = 1$; second, the surface $(x/a)^{\frac{2}{3}} + (y/b)^{\frac{2}{3}} + (z/c)^{\frac{2}{3}} = 1$ $(a > b > c)$.

Digitized by Google

Solution by the PROPOSER.

(i.) Let $2x$, $2y$ be the sides of the rectangle. Then

$$xy = \text{maximum} \quad \ldots\ldots(1),$$

$$(x/a)^{\frac{1}{2}} + (y/b)^{\frac{1}{2}} = 1 \quad \ldots\ldots(2).$$

From (1), $dy/dx = -y/x$; from (2), $dy/dx = -b^{\frac{1}{2}}y^{\frac{1}{2}}/(a^{\frac{1}{2}}x^{\frac{1}{2}})$; therefore $(y/b)^{\frac{1}{2}} = (x/a)^{\frac{1}{2}}$, which in (2) gives $x = a/(2\sqrt{2})$, $y = b/(2\sqrt{2})$.

(ii.) Let $2x$, $2y$, $2z$ be the edges of the parallelopiped. Then

$$xyz = \text{maximum} \quad \ldots\ldots(3),$$

$$(x/a)^{\frac{1}{2}} + (y/b)^{\frac{1}{2}} + (z/c)^{\frac{1}{2}} = 1 \quad \ldots\ldots(4).$$

From (3), $dz/dx = -z/x$, $dz/dy = -z/y$;

from (4), $dz/dx = -c^{\frac{1}{2}}z^{\frac{1}{2}}/(a^{\frac{1}{2}}x^{\frac{1}{2}})$, $dz/dy = -c^{\frac{1}{2}}z^{\frac{1}{2}}/(b^{\frac{1}{2}}y^{\frac{1}{2}})$;

therefore $$(x/a)^{\frac{1}{2}} = (y/b)^{\frac{1}{2}} = (z/c)^{\frac{1}{2}} \quad \ldots\ldots(5).$$

Hence, from (4) and (5), $x = a/(3\sqrt{3})$, $y = b/(3\sqrt{3})$, $z = c/(3\sqrt{3})$.

15441 & 15014. (Professor NANSON.)—(15441.) Three triangles Δ_1, Δ_2, Δ_3 are inscribed in a conic. If Δ_2, Δ_3 are co-polar with Δ_1, then so is the triangle formed by the joins of corresponding vertices of Δ_2, Δ_3.

(15014.) A conic touches BC, CA, AB in D, E, F. If DO, EO, FO cut the conic again in P, Q, R, prove AP, BQ, CR concurrent.

Solutions by the PROPOSER.

Take Δ_1 as the triangle of reference; then, if the conic is

$$f/x + g/y + h/z = 0,$$

the join of corresponding vertices (x', y', z'), (x'', y'', z'') of Δ_2, Δ_3 is

$$fx/(x'x'') + gy/(y'y'') + hz/(z'z'') = 0,$$

and at the intersections of this with $x = 0$ we have

$$gy/(y'y'') + hz/(z'z'') = 0 \quad \ldots\ldots(1).$$

But, if (p', q', r'), (p'', q'', r'') are the "poles" of $\Delta_1\Delta_2$, $\Delta_1\Delta_3$, we have $y'/q' = z'/r'$, $y''/q'' = z''/r''$. Thus (1) becomes $gy/(q'q'') + hz/(r'r'') = 0$. Hence Δ and the triangle formed by the joins of corresponding vertices of Δ_2, Δ_3 have the "axis" $fx/(p'p'') + gy/(q'q'') + hz/(r'r'') = 0$, and are therefore "co-polar."

Making Δ_2, Δ_3 coincide, we get Question 15014.

In the solution of Question 15441 published in Vol. VI. (New Series), p. 93, the meaning of "co-polar" is missed, and consequently Δ_2, Δ_3 coincide with the triangle formed by the tangents at the vertices of Δ so that nothing is proved.

12518. (Professor RUSSO.)—On donne, sur un même plan, une droite Δ et un cercle Γ fixes; soit Γ' un second cercle mobile, de rayon invariable, tangent à Δ. Trouver (1) l'enveloppe de l'axe radical δ des cir-

Digitized by Google

conférences Γ et Γ′; et (2) le lieu du point de rencontre de δ avec la ligne des centres des circonférences, Γ, Γ′.

Solution by W. J. Greenstreet, M.A.

Take as axes the parallel to Δ through B, the centre of Γ′, and the perpendicular to Δ through A, the centre of Γ. Let these meet in O, and take OA = a, OB = b, and let the radii be r_a, r_b. Then we easily find the the radical axis to be

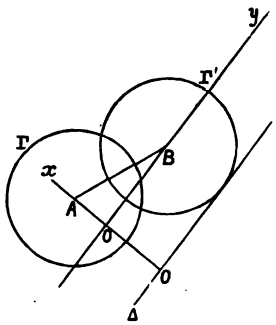
$$2by - 2ax = r_a^2 - r_b^2 + b^2 - a^2 \ldots \quad (\delta)$$

and AB is $ay + bx = ab$. The envelope of the radical axis

$$b^2 - 2by + (2ax - a^2 + r_a^2 - r_b^2) = 0$$

is $$y^2 = 2ax + r_a^2 - r_b^2 - a^2 \ldots \quad (1),$$

a parabola, focus $(\frac{1}{2}a, 0)$, the mid-point of OA. The intersection of (δ) and AB is the pedal of (1) with respect to A, viz.,



$$ay^2(a - 2x) + (a - x)^2(2ax - r_a^2 + r_b^2 + a^2) = 0,$$

a circular cubic.

15518. (James Blaikie, M.A.)—A pack of 52 cards is shuffled in the following way:—The 2nd card is placed over the 1st, the 3rd under the 1st, the 4th over the 2nd, the 5th under the 3rd, and so on alternately. Find after how many such shuffles the pack will have regained its original order. Solve a similar question for packs of 10, 24, 30, 32 cards, and establish any general principles.

Note by Professor W. H. H. Hudson, M.A.

This was practically set at St. John's College, Cambridge, June 2, 1863, in this form:—"A pack of 52 cards is shuffled in such a manner that the new pack is formed by taking the cards in order from the old pack and putting them alternately at the top and bottom of the new pack. Show that after 9 such operations every 13th card will be together, and that after 3 more such operations the cards will be in the same order as at first. Also give a rule for determining the number of such operations required in order to bring a pack of n cards again to their original order."

In connexion with this I quote also *Educational Times*, January, 1865, Question 1635:—"A pack of N cards is shuffled in any manner whatever, and then again in a precisely similar manner, and so on; show how to find after how many shufflings at most the cards return to their original position. If N = 52, the utmost number of shufflings is 180180."

The solution is given in the *Reprint*, Vol. III., p. 105.

The Proposer contributes the following discussion:—If we designate the 52 cards by the numbers 1, 2, 3, ..., 51, 52, and write under these

Digitized by Google

numbers the numbers of the cards which will fall into these places after the first shuffle, viz., 52, 50, ..., 49, 51, we find that 52 falls under 1, and, looking to 52 in the top line, we see that 51 falls under it, 49 under 51, &c. Thus the 1st place is held in turn by the cards numbered 1, 52, 51, 49, 45, 37, 21, 12, 30, 7, 40, 27, 1, and the 1st card returns to its original place after 12 shuffles. Similarly it is found that each other card returns to its original place after a number of shuffles which is either 12, 6, 4, 3, or 1. Thus all the cards are in their 1st places after 12. Similarly we can show that with packs of 10, 24, and 30 cards the numbers are 6, 21, and 30 shuffles. For 32 the number is 6, and it can be shown that for 2^n cards the number is $n+1$.

15519. (Professor SANJÂNA, M.A.)—Find the condition that

$$x^9 + ax^7 + bx^5 + cx^3 + d$$

may be divisible by $(x+h)^3$. Also determine p, q, r in order that

$$x^9 + px^7 + qx^5 + rx^3 + 1$$

may be divisible by $(x-1)^3$ or $(x+1)^3$.

Solutions (I.) *by* Lt.-Col. ALLAN CUNNINGHAM, R.E.;
(II.) *by* JAMES BLAIKIE, M.A., *and* C. M. ROSS.

(I.) (i.) All that is necessary is that the original equation, say $F(x) = 0$, and its first and second derived equations, $F'(x) = 0$, $F''(x) = 0$, should have common roots. Forming $F'(x)$, and dividing through by x^2, gives

$$F(x) = x^9 + ax^7 + bx^5 + cx^3 + d = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$F'(x) = 9x^6 + 7ax^4 + 5bx^2 + 3c = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Now form $F''(x)$, and divide through by $2x$, giving

$$F''(x) = 27x^4 + 14ax^2 + 5b = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Taking the difference $3F'(x) - x^2 . F''(x)$ gives

$$7ax^4 + 10bx^2 + 9c = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(4).$$

Eliminating x^4 between (3), (4) gives

$$(270b - 98a^2)x^2 + (243c - 35ab) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(5).$$

This gives a value of x^2 (or two of x) common to (2), (3). Substituting in (4) gives the condition that (2), (3) may have common roots, viz.,

$$7a(243c - 35ab)^2 - 10b(243c - 35ab)(270b - 98a^2) + (270b - 98a^2)^2 = 0 \ldots(6).$$

Next, taking the difference $9F(x) - x^3 . F'(x)$ gives

$$2ax^7 + 4bx^5 + 6cx^3 + 9d = 0 \ldots\ldots\ldots\ldots\ldots\ldots(7).$$

Also, taking the difference $7 \times$ equation (7) $- 2x^3 \times$ equation (4) gives

$$8bx^5 + 24cx^3 + 63d = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(8).$$

Next, taking the difference $7a \times$ equation (8) $- 8b \times$ equation (4) gives

$$(168ac - 80b^2)x^3 - 72bcx + 441ad = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(9).$$

Substituting the value of x^3 from (5) into (9) gives

$$[(168ac - 80b^2)(243c - 35ab) + 72bc(270b - 98a^2)]x = 441ad(270b - 98a^2)$$
$$\ldots\ldots\ldots\ldots\ldots\ldots(10),$$

Digitized by Google

and the (single) value of x in equation (10) is the root common to equations (1), (2), (3), and this value of x substituted into equation (5), or indeed into any of the preceding equations, gives the second condition required [in addition to (6)] that (1), (2), (3) may have common roots. When these two conditions are satisfied, then substituting $x = -h$ in (10) gives the further condition that $(x+h)^3$ may be a divisor of $F(x)$.

(ii.) Writing $a = p$, $b = q$, $c = r$, $d = 1$ in equations (1), (2), (3) above, the conditions that $f(x) = x^9 + px^7 + qx^5 + rx^3 + 1$ may be divisible by $(x-1)^3$ are found by writing $x = +1$ in the modified equations (1), (2), (3), giving

$$p+q+r = -2, \quad 7p+5q+3r = -9, \quad 14p+5q = -27 \ldots\ldots(11);$$

whence $$p = -\tfrac{29}{6}, \quad q = \tfrac{23}{6}, \quad r = -\tfrac{43}{6} \ldots\ldots\ldots\ldots\ldots\ldots(12);$$

and the conditions that $f(x)$ may be divisible by $(x+1)^3$ are found by writing $x = -1$ in the modified equations (1), (2), (3), giving

$$p+q+r = 0, \quad 7p+5q+3r = -9, \quad 14p+5q = -27 \ldots\ldots (13);$$

whence $$p = -\tfrac{9}{4}, \quad q = -\tfrac{9}{4}, \quad r = \tfrac{27}{6} \ldots\ldots\ldots\ldots\ldots\ldots(14).$$

(II.) If $f(x) \equiv x^9 + ax^7 + bx^5 + cx^3 + d$ be divisible by $(x+h)^3$, then we obtain the three equations $f(-h) = 0$, $f'(-h) = 0$, $f''(-h) = 0$, or

$$\left.\begin{aligned} h^9 + \quad ah^7 + \quad bh^5 + ch^3 - d &= 0 \\ 9h^6 + \quad 7ah^4 + \quad 5bh^2 + 3c \quad &= 0 \\ 36h^6 + 21ah^4 + 10bh^2 + 3c \quad &= 0 \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots(a).$$

The eliminant of these three equations can be expressed as a determinant of the eleventh order and gives the required condition.

If $h = -1$, $a = p$, $b = q$, $c = r$, $d = 1$, equation (a) becomes

$$\left.\begin{aligned} p + \quad q + \quad r + \quad 2 &= 0 \\ 7p + \quad 5q + 3r + \quad 9 &= 0 \\ 21p + 10q + 3r + 36 &= 0 \end{aligned}\right\},$$

giving $p = -\tfrac{29}{6}$, $q = \tfrac{23}{4}$, $r = -\tfrac{43}{6}$, giving the relation

$$8x^9 - 39x^7 + 66x^5 - 43x^3 + 8$$
$$= (x-1)^3 (8x^6 + 24x^5 + 9x^4 - 37x^3 - 48x^2 - 24x - 8).$$

When $h = 1$, the first equation is $p+q+r = 0$, and we obtain the values $p = -\tfrac{9}{4}$, $q = -\tfrac{9}{4}$, $r = \tfrac{27}{6}$, giving the relation

$$8x^9 - 9x^7 - 18x^5 + 27x^3 + 8$$
$$= (x+1)^3 (8x^6 - 24x^5 + 39x^4 - 53x^3 + 48x^2 - 24x + 8).$$

15574. (R. W. D. Christie.)—Find an infinite number of positive integral values of X *having the same* Y in the Pellian equation

$$X_n^2 - PY^2 = -1;$$

e.g., $$70^2 - 29 \cdot 13^2 = -1, \quad 746^2 - 3293 \cdot 13^2 = -1, \ldots.$$

Solutions (I.) *by* Lt.-Col. Allan Cunningham, R.E.; (II.) *by the* Proposer.

(I.) The only condition necessary is

$$X_n^2 \equiv -1 \pmod{Y^2} \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Now, if Y be given, let X_1 be the lowest value of X_n satisfying the con-

Digitized by Google

gruence; then $X_1' = Y^2 - X_1$ is the other solution less than Y^2; and the general values are $X_n = nY^2 + X_1$, $X_n' = nY^2 + X_1'$ .....(2), and the values of P corresponding are

$$P_n = (X_n^2 + 1)/Y, \quad P_n' = (X_n'^2 + 1)/Y \quad \text{.....(3).}$$

Col. Cunningham's paper on "Period Lengths of Circulates" in the *Mess. of Math.*, Vol. XXIX., gives a general direct method by which the lowest pair of solutions (X_1, X_1') can always be found when a solution (x) of $x^2 \equiv -1 \pmod{Y}$ is known, and gives also a table (p. 166) of these solutions (X_1, X_1') for all prime moduli (X_1) up to 113; this has been since greatly extended (in MS.) by the author.

Ex.—The roots given (p. 166) for the modulus $Y = 13^2$ are $X_1 = 70$, $X_1' = 99$; hence the general values are $X_n = n.169 + 70$, $X_n' = n.169 + 99$, with P_n, P_n' as in (3).

(II.) (i.) We have

$$[(a^2+b^2)q + ac + bd]^2 + (ad-bc)^2 = (a^2+b^2)[(aq+c)^2 + (bq+d)^2]$$
$$= [a(aq+c) \pm b(bq+d)]^2 + [b(aq+c) \mp a(bq+d)]^2,$$

and a sign may be changed. Consequently in the equation requiring same values of Y, viz., $X^2 - PY^2 = -1$, we can at our discretion make either P or $Y = a^2 + b^2$ or $(aq \pm c)^2 + (bq \mp d)^2$, where a, b, c, d are known or found from P and q is arbitrary, to find a suitable value making the two squares equal to a square.

In (i.) then let $a^2 + b^2 = Y^2 =$ a square, $ad - bc = 1$ as usual where a and b are given, say $a = 5$, $b = 12$, $Y = 13$; then $c = 2$, $d = 5$. Thus for $q_1 = 0, 1, 2, \ldots$ we have, by (i.), with infinite series for each q, *i.e.*,

$$70^2 + 1^2 = 29 \cdot 13^2, \qquad 99^2 + 1^2 = 29 \cdot 338,$$

and generally

$$29q \text{ or } (13^2k + 70)^2 + 1 = 13^2Y \ (k \text{ arbitrary} + 70)^2 + 1 = 29 \cdot Y,$$

where q is arbitrary, ...;

$q = 1$: $239^2 + 1^2 = 338\,(13)^2 = 2 \cdot 13^4$,

$(338k + 239)^2 + 1^2 = 338 \cdot Y = 2Y\,(13^2)$, ...;

$= 2$: $408^2 + 1^2 = 985\,(13)^2$, $1393^2 + 1^2 = 985 \cdot Y$, ...

Incidentally it may be mentioned that the above values for X and P may be got from $x^2 - 2y^2 = 1$ when Y is constant.

(ii.) A somewhat different, but equally general, solution is obtainable as follows:—

Let $b(AB) - 2a(A + B) = 1$. Take $A = 1$, $B = 5$ (any two odd numbers to secure integers). Thus $5b - 12a = 1$. Here

$$a_n = 5k + 2 = 2, 7, 12, \ldots; \quad b_n = 12k + 5 = 5, 17, 29, \ldots.$$

Thus

$$70^2 - (5^2 + 2^2)\,13^2 = -1,$$
$$239^2 - (17^2 + 7^2)\,13^2 = -1 \quad \text{or} \quad 239^2 - 2 \cdot 13^4 = -1,$$
$$408^2 - (29^2 + 12^2)\,13^2 = -1, \ \ldots, \text{ ad inf.}$$

Combining both methods, it is interesting to note that we may secure

$$X_1^2 - P_1Y^2 = -1, \quad X_2^2 - 2P_1Y^2 = -1,$$

a new result, which may be generalized.

Digitized by Google

15500 & 15572. (Professor E. B. Escott.)—(15500.) Find two large consecutive numbers ($>10^{14}$) all of whose prime factors are less than 100.

Examples: $a^2 = 7534241^2 = 11^2 . 13^2 . 19^2 . 47^2 . 59^2$,
$a^2-1 = 2^6 . 3^3 . 5 . 7^3 . 31^2 . 41^2 . 83$;

$a^2 = 154738092 = 13^4 . 19^2 . 61^2 . 79^2$,
$a^2-1 = 2^5 . 3^3 . 5 . 7^2 . 11 . 17 . 41 . 43 . 47 . 73$.

(15572.) Find two large consecutive numbers ($>10^{17}$) all of whose prime factors are less than 200. Examples:

$a^2 = 306 . 331 . 585^2 = 5^2 . 7^3 . 17^2 . 19^2 . 79^2$,
$a^2-1 = 2^7 . 3 . 11^2 . 13 . 31 . 37 . 83 . 101 . 107 . 151$;

$a^2 = 315 . 630 . 756^2 = 17766^4 = 2^4 . 3^{12} . 7^4 . 47^4$,
$a^2-1 = 5 . 11 . 13 . 17 . 19 . 37 . 73 . 89 . 101 . 109 . 163$;

$a^2 = 940 . 737006^2 = 2^2 . 3^4 . 11^4 . 19^2 . 127^2 . 179^2$,
$a^2-1 = 5 . 7 . 13 . 17 . 23 . 31 . 41 . 47 . 59 . 71 . 103 . 193$.

Discussion of the more general Problem by the Proposer.

The Proposer discusses the general methods applicable to the solution of these problems as follows:—

In the computation of logarithms by means of the series

$$\log(x+a) = \log x + 2m\left[\frac{a}{2x+a} + \frac{1}{3}\left(\frac{a}{2x+a}\right)^3 + \dots\right]$$

it is important that we be able to determine two numbers $x+a$ and x all of whose factors are not greater than the number whose logarithm is desired. a should be small in comparison with x (preferably $a=1$). Huygens (*Opera varia*, Vol. II., pp. 456–458) and Gauss (*Werke*, Band II., Seite 501–3) have given lists of such pairs of numbers suitable for calculating logarithms of numbers as far as 100.

The method I have used in determining these numbers is as follows. By methods which I shall give in a paper soon to be published I have obtained the following pairs of polynomials which have rational linear factors and which differ only in their constant terms. Some of these were given by Borda, *Logarithmic Tables;* Lavernède, *Gard. Notice Trav. Acad.*, 1807, pp. 179–192; Gergonne's *Annales de Math.*, Vol. I., 1810–11, pp. 18–51, 78–100; Lacroix, *Traité du calcul diff. et int.*, tome I., 1810, pp. 48–52; Secretan, *Comptes rendus*, t. XLIV., 1857, pp. 1276–9; Thomas Knight, *Phil. Trans. Roy. Soc. of London*, 1817, pp. 217–233; Wm. Allman, *Trans. Royal Irish Acad.*, Vol. VI., 1797, pp. 391–434.

i. $x(x-a-b) = x^2-(a+b)x$,
$(x-a)(x-b) = x^2-(a+b)x+ab$.

ii. In the preceding, let $b=-a$:
$x^2 = x . x$,
$x^2-a^2 = (x+a)(x-a)$.

iii. $(x+a+b)(x-a)(x-b) = x^3-(a^2+ab+b^2)x+ab(a+b)$,
$(x-a-b)(x+a)(x+b) = x^3-(a^2+ab+b^2)x-ab(a+b)$.

Digitized by Google

iv. In the last, let $a = b = 1$:
$(x+2)(x-1)^2 = x^3-3x+2,$
$(x-2)(x+1)^2 = x^3-3x-2.$

v. $(x-3)(x-5)(x+8) = x^3-49x+120,$
$(x+7)\,x\,(x-7) \quad = x^3-49x.$

vi. $(x+11)(x-5)(x-6) = x^3-91x+330,$
$(x+9)(x+1)(x-10) = x^3-91x-90.$

vii. $(x+2)(x+11)(x-13) = x^3-147x-286,$
$(x+7)^2(x-14) \quad = x^3-147x-686.$

viii. $(x+4)(x-1)(x-4) = x^3-x^2-16x+16,$
$(x+2)^2(x-5) \quad = x^3-x^2-16x-20.$

ix. $(x+7)\,x\,(x-8) = x^3-x^2-56x,$
$(x+4)^2(x-9) \quad = x^3-x^2-56x-144.$

x. $x^2(x+5)(x-5) \quad = x^4-25x^2,$
$(x+4)(x+3)(x-3)(x-4) = x^4-25x^2+144.$

xi. $x^2(x-5)^2 \quad = x^4-10x^3+25x^2,$
$(x+1)(x-2)(x-3)(x-6) = x^4-10x^3+25x^2-36.$

xii. $x^2(x-13)^2 \quad = x^4-26x^3+169x^2,$
$(x+2)(x-3)(x-10)(x-15) = x^4-26x^3+169x^2-900.$

xiii. $(x+7)(x-7)(x+1)(x-1) = x^4-50x^2+49$ } (Practically same
$(x+5)^2(x-5)^2 \quad = x^4-50x^2+625$ } as xi.)

xiv. $(x+5)(x+1)(x-2)(x-6) = x^4-2x^3-31x^2+32x+60,$
$(x+4)(x+3)(x-4)(x-5) = x^4-2x^3-31x^2+32x+240.$

xv. $(x+9)(x+1)(x-8)(x+5)(x-7) = x^5-110x^3+2629x+2520,$
$(x-9)(x-1)(x+8)(x-5)(x+7) = x^5-110x^3+2629x-2520.$

xvi. $(x-10)(x+9)(x+7)(x-2)(x-4) = x^5-125x^3+3004x-5040,$
$(x+10)(x-9)(x-7)(x+2)(x+4) = x^5-125x^3+3004x+5040.$

xvii. $x^2(x-7)^2(x+7)^2 \quad = x^6-98x^4+2401x^2,$
$(x+8)(x+5)(x+3)(x-3)(x-5)(x-8) = x^6-98x^4+2401x^2-14400.$

xviii. $(x+10)(x-10)(x+9)(x-9)(x+1)(x-1) =$
$(x+11)(x-11)(x+6)(x-6)(x+5)(x-5) =$

xix. $(x+12)(x-12)(x+11)(x-11)(x+1)(x-1) =$
$(x+13)(x-13)(x+9)(x-9)(x+4)(x-4) \quad =$

xx. $x^2(x+13)^2(x-13)^2 \quad =$
$(x+15)(x-15)(x+8)(x-8)(x+7)(x-7) =$

xxi. $x^2(x+19)^2(x-19)^2 \quad = (x^3-361x)^2,$
$(x+21)(x+16)(x+5)(x-5)(x-16)(x-21) =$

xxii. $x^2(x+8)^2(x-8)^2(x+11)^2(x-11)^2 = (x^5-185x^3+7744x)^2,$
$(x^2+2x-108)(x^2-2x-108)(x+10)(x-10)(x+7)(x-7)$
$\times (x+1)(x-1) =$

xxiii. $(x+29)^2(x+17)^2x^2(x-17)^2(x-29)^2 =$
$(x^2+9x-640)(x^2-9x-640)(x+27)(x-27)(x+11)(x-11)$
$\times (x+7)(x-7) =$

Digitized by Google

xxiv. $(x+31)^2(x+19)^2(x-19)^2(x-31)^2x^8 =$
$(x^2+5x-896)(x^2-5x-896)(x+25)(x-25)(x+11)(x-11)$
$\times(x+9)(x-9) =$

xxv. $(x+68)^2(x+57)^2x^2(x-57)^2(x-68)^2 =$
$(x^2+130x+4356)(x^2-130x+4356)(x+75)(x+42)(x+13)(x-13)$
$\times(x-42)(x-75).$

As an example of the method, take the formula

xvii. $$\frac{x^2(x-7)^2(x+7)^2}{(x+8)(x+5)(x+3)(x-3)(x-5)(x-8)} = \frac{x^6-98x^4+2401x^2}{x^6-98x^4+2401x^2-14400} \quad \ldots\ldots\ldots\ldots\ldots(1).$$

The numerator and denominator differ by 14400. If we choose x so that *either* numerator or denominator is divisible by 14400, then the other one will also be divisible, and after dividing both terms of the fraction by 14400 the numerator and denominator will differ by unity. Then x must satisfy the congruence $x^2(x-7)^2(x+7)^2 \equiv 0 \pmod{14400 = 2^6 \cdot 3^2 \cdot 5^2}$. The solution is

$$x \equiv 3, 5, 7, 13, 15, 17 \pmod{20} \quad \text{or} \quad x \equiv 0, 8, 32 \pmod{40}.$$

For example, let $x = 253$. Substituting in (1) and reducing to its lowest terms, $$\frac{11^2 \cdot 13^2 \cdot 23^2 \cdot 41^2}{2^7 \cdot 3 \cdot 5^2 \cdot 7^2 \cdot 29 \cdot 31 \cdot 43} = \frac{18184252801}{18184252800}.$$

[* Denotes numerator less 1.]

If $x = 920$, $$\frac{3^2 \cdot 11^2 \cdot 23^2 \cdot 83^2 \cdot 103^2}{2^3 \cdot 5 \cdot 7 \cdot 13 \cdot 19 \cdot 29 \cdot 37 \cdot 61 \cdot 71 \cdot 131} = \frac{6488691^2}{*}.$$

If $x = 3325$,
$$\frac{5^2 \cdot 7^8 \cdot 17^2 \cdot 19^2 \cdot 79^2}{2^7 \cdot 3 \cdot 11^2 \cdot 13 \cdot 31 \cdot 37 \cdot 83 \cdot 101 \cdot 107 \cdot 151} = \frac{306331585^2}{*}.$$

If $x = 4833$,
$$\frac{2^2 \cdot 3^4 \cdot 11^4 \cdot 19^2 \cdot 127^2 \cdot 179^2}{5 \cdot 7 \cdot 13 \cdot 17 \cdot 23 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71 \cdot 103 \cdot 193} = \frac{940737006^2}{*}.$$

In xviii. let $x = 1639$:
$$\frac{2^3 \cdot 3^2 \cdot 13 \cdot 17 \cdot 41 \cdot 97 \cdot 103 \cdot 163 \cdot 181}{5 \cdot 11^2 \cdot 19 \cdot 23 \cdot 37 \cdot 43 \cdot 47 \cdot 71 \cdot 137} = \frac{192301984069416}{192301984069415}.$$

In xx. let $x = 3835$:
$$\frac{2^2 \cdot 7^2 \cdot 13^6 \cdot 37^2 \cdot 59^2}{3 \cdot 5 \cdot 11^2 \cdot 17 \cdot 29 \cdot 43 \cdot 61 \cdot 89 \cdot 113 \cdot 191} = \frac{67144714^2}{*}.$$

In xi. let $x = 9638$:
$$\frac{13^4 \cdot 19^2 \cdot 61^2 \cdot 79^2}{2^5 \cdot 3^3 \cdot 5 \cdot 7^2 \cdot 11 \cdot 17 \cdot 41 \cdot 43 \cdot 47 \cdot 73} = \frac{15473809^2}{*}.$$

In iv. let $x = 28898$:
$$\frac{5^2 \cdot 11^2 \cdot 17^2 \cdot 37^2 \cdot 71^2}{2^3 \cdot 3^5 \cdot 7 \cdot 13^4 \cdot 19^2 \cdot 43} = \frac{4912490^2}{*}.$$

In finding values of x so that all the factors in these polynomials shall be small it is of very great help, or, in fact, is essential, to have factor tables giving all factors of all numbers. The only table of this kind of which I know is Edward Hinkley's *Tables of the Prime Numbers and Prime Factors of the Composite Numbers* (Baltimore, 1853), which gives all the

Digitized by Google

factors of odd numbers to 20000 and even numbers to 12500. For instance, the factors of 19635 are given as $3 . 5 . 7 . 11 . 17$.

There are a few special methods which I have found useful in connexion with the above. When we have the two fractions $(x+1)/x$ and $(y+1)/y$, the quotient of the two, $[(x+1)/x][y/(y+1)] = (xy+y)/(xy+x)$, will have its numerator and denominator differing by $y-x$. Therefore, if the numerators and denominators of two fractions of this form have common factors whose product is $y-x$, then the quotient of these two fractions will be another fraction of the same form.

Example : $\frac{3^8 . 11^4}{2^4 . 5^2 . 7^2 . 13^2 . 29} = \frac{9801^2}{*}$, $\frac{3^4 . 19^4}{2^5 . 5^3 . 7 . 13 . 29} = \frac{3249^2}{*}$,

$$3^8 . 11^4 - 3^4 . 19^4 = 3^4 (33^4 - 19^4) = 3^4 (33^2 + 19^2)(33 + 10)(33 - 19)$$
$$= 2^4 . 3^4 . 5^2 . 7 . 13 . 29.$$

As these factors are all common to the numerators or denominators of the two fractions, their quotient

$$\frac{7 . 13 . 19^4}{2 . 3^4 . 5 . 11^4} = \frac{11859211}{11859210}$$

has its denominator less by a unit than its numerator.

If the difference between y and x is small compared with either x or y, the quotient of the two fractions will give a fraction whose terms will be much larger numbers than the first fractions.

Examples :

1. $\frac{2^2 . 3^2 . 7^2 . 31^2 . 97^2}{5 . 13 . 17^2 . 19 . 23 . 29 . 67} = \frac{126294^2}{*}$,

$$\frac{5^2 . 7^2 . 41^2 . 97^2}{2^2 . 3^2 . 11 . 17 . 19 . 23 . 37 . 89} = \frac{139195^2}{*}.$$

Dividing first by second and simplifying, we have

$$\frac{2^5 . 3^4 . 11 . 31^2 . 37 . 89}{5^2 . 13 . 17 . 29 . 41^2 . 67} = \frac{90228305376}{*}.$$

2. $\frac{3^8 . 5^4 \ 17^2 . 29^2}{2^2 . 7^2 . 31 . 61 . 83 . 97 . 167} = \frac{998325^2}{*}$,

$$\frac{3^4 . 5^4 . 7^2 . 13^2 . 29^2}{2^5 . 17 . 31 . 37 . 59 . 61 . 157} = \frac{593775^2}{*}.$$

Dividing second fraction by first,

$$\frac{7^4 . 13^2 . 83 . 97 . 167}{2^2 . 3^4 . 17^2 . 37 . 59 . 157} = \frac{545563318573}{*}.$$

3. In xi. let $x = 13208$:

$$\frac{2^4 . 3^6 . 13^2 . 127^2 . 163^2}{5 . 7^2 . 17 . 19 . 23 . 31 . 37 . 41 . 71 . 139} = \frac{29064204^2}{*}.$$

In xxi. let $x = 3605$:

$$\frac{11^2 . 103^2 . 151^2 . 163^2}{2^7 . 3 . 5 . 7 . 17 . 19^2 . 37^2 . 71 . 97} = \frac{27886529^2}{*}.$$

The numerators and denominators of the fractions have as common factors all the factors of $y-x$ excepting 5 and 17. Therefore when we divide one by the other and reduce to lowest terms the new fraction will have its terms differing by 85, *i.e.*,

$$\frac{7 . 11^2 . 23 . 31 . 41 . 103^2 . 139 . 151^2}{2^{11} . 3^7 . 13^2 . 19 . 37 . 97 . 127^2} = \frac{832530093941385301}{832530093941385216}.$$

Digitized by Google

The following method is also useful:—If the fraction a/b be expressed as a continued fraction, and if k/l be the convergent immediately preceding a/b, then $(al)/(bk)$ will have its numerator and denominator differing by unity. More generally, if k/l and a/b are two consecutive convergents to a continued fraction, the fractions

$$[a(rb \pm l)]/[b(ra \pm k)] \quad \text{and} \quad [l(rk \pm a)]/[k(rl \pm b)]$$

will have their terms differing by unity. Example:

$$\frac{2^2 \cdot 5^2 \cdot 31^2}{19^2 \cdot 13^2} = \frac{96100}{61009} = (1, 1, 1, 2, 1, 4, 1, 2, 1, 2, 3, 3, 1, 1, 1, 3)$$

$$= 1 + \frac{1}{1+} \frac{1}{1+} \frac{1}{2+} \frac{1}{1+} \frac{1}{4+} \frac{1}{1+} \frac{1}{2+} \frac{1}{1+} \frac{1}{2+} \frac{1}{3+} \frac{1}{3+} \frac{1}{1+} \frac{1}{1+} \frac{1}{1+} \frac{1}{3}.$$

The last two convergents are $\frac{26411}{16767}$ and $\frac{96100}{61009}$. Then

$$\frac{96100 \cdot 16767}{61009 \cdot 26411} = \frac{2^2 \cdot 3^6 \cdot 5^2 \cdot 23 \cdot 31^2}{7^4 \cdot 11 \cdot 13^2 \cdot 19^2} = \frac{1611308700}{1611308699}.$$

In formula iv., $[(x+2)(x-1)^2]/[(x+1)^2(x-2)]$, put $x-2 = a^2$. After dividing numerator and denominator by 4 they will differ by unity, and the fraction will be of the form $(b^2+1)/b^2$.

Example.—Let $a = 237$, $x = 56171$:

$$\frac{5^2 \cdot 13 \cdot 29 \cdot 41^2 \cdot 137^2 \cdot 149}{2^2 \cdot 3^4 \cdot 31^2 \cdot 79^2 \cdot 151^2} = \frac{6656382^2 + 1}{6656382^2}.$$

From Gauss's table of factors of a^2+1 (*Werke*, Bd. II.), find numbers a whose factors are small and where the factors of a^2+1 are small. Then, if the factors of $a+1$ and $a-1$ are likewise small, the expression

$$a^4/(a^4-1) = a^4/[(a^2+1)(a+1)(a-1)]$$

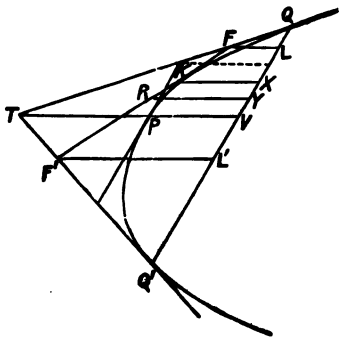
will give further examples.

Example.—$a = 17766$:

$$\frac{a^4}{a^4-1} = \frac{2^4 \cdot 3^{12} \cdot 7^4 \cdot 47^4}{5 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 37 \cdot 73 \cdot 89 \cdot 101 \cdot 109 \cdot 163} = \frac{315630756^2}{*}.$$

15599. (A. M. Nesbitt, M.A.) —Two tangents drawn from a point T to a parabola meet any third tangent in F and F′. If the diameter through T meet the curve in P, prove that the tangent at P bisects FF′.

[Solution by elementary geometrical conics required.]



Digitized by Google

Solutions (I.) *by* R. F. Davis, M.A.; (II.) *by* W. F. Beard, M.A., *and the* Proposer; (III.) *by* Kela Deva Rau, B.A., *and others.*

(I.) Since $QF : QT = QL : QV = 2QL : 2QV = QX : QQ'$, therefore FX is parallel to TQ'. Similarly, F'X is parallel to TQ. Hence TFXF' is a parallelogram, and the middle point of one diagonal FF' coincides with the middle point of the other diagonal TX. But this latter point evidently lies on the tangent at P, &c. (See Milne and Davis, 1890, p. 25.)

(II.) Let FF' touch the parabola at R and meet the tangent at P in G. Draw GM, RN parallel to the axis,

$NM = \frac{1}{2}VN$, $NL' = \frac{1}{2}NQ'$;

therefore $L'M = \frac{1}{2}Q'V$;

also $NL = \frac{1}{2}NQ$;

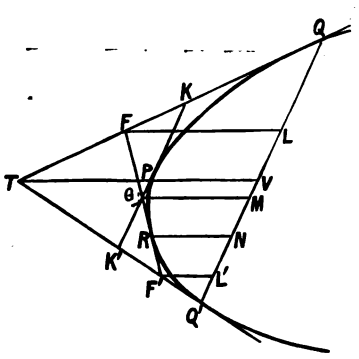
therefore

$LL' = \frac{1}{2}QQ' = Q'V$;

therefore

$LL' = 2L'M$;

therefore M is mid-point of LL'; therefore G is mid-point of FF'.



(III.) Let the tangent drawn from P meet TQ at K and FF' at R. From F, K, R, T, F' draw FL, KX, RY, TV, and F'L' parallel to the axis. From a well known property of the parabola, PQ is bisected by KX; therefore VQ = 2VX. Similarly Q'X = 2Q'L', Q'Q = 2Q'V; therefore QQ' − Q'X = 2(Q'V − Q'L'), *i.e.*, XQ = 2L'V. But XQ = 2XL; therefore LX = L'V, and XY = VY; therefore LY = YL'; therefore FR = RF', *i.e.*, FF' is bisected by the tangent at P.

15407. ("Stranger.")—Find a point on a given parabola which is the centre of a circle touching the curve and passing through its focus.

Solutions (I.) *by* R. Tucker, M.A.; (II.) *by* Professor Sanjána, M.A.; (III.) *by* Lionel E. Reay, B.A.

(I.) The equation to the circle touching the parabola at $(am^2, 2am)$ and passing through the focus is

$$x^2 + y^2 - ax(3m^2 + 1) - ay(3m - m^3) + 3a^2m^2 = 0.$$

Hence the co-ordinates of the centre are $2x = a(3m^2 + 1)$, $2y = a(3m - m^3)$, and, that this may be on the curve, $(3m - m^3)^2 = 8(3m^2 + 1)$, *i.e.*,

$$m^6 - 6m^4 - 15m^2 - 8 = 0 = (m^2 + 1)^2(m^2 - 8).$$

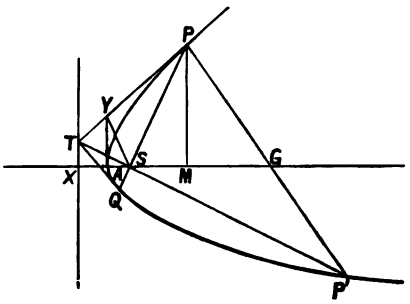
Digitized by Google

Therefore $2x = a\,(25)$, $2y = a\,.\,5\,(2\sqrt{2})$.

(II.) Let the normal at P meet the curve in P′, and suppose P′S = P′P. Then the triangles SPG, SPP′ are similar, and

$$SP^2 = PG\,.\,PP'.$$

If PS meets the curve again in Q, the tangent at Q meets PT on the directrix and is parallel to PP′; hence PP′ = 4TQ; also PG = 2SY. Thus



$$SP^2 = 8SY\,.\,TQ = 8SY^2\,(TQ/SY) = 8SY^2\,(TP/YP) = (8SY^2/YP^2)\,SP^2,$$

so that $PY^2 = 8SY^2$ and $SP^2 = 9SY^2 = 9SP\,.\,SA$. Hence, finally, SP = 9SA, or the abscissa of P with reference to the directrix is 9SA; therefore P is known, and P′ is the extremity of the normal chord at P.

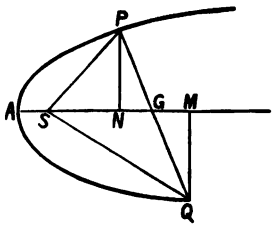
(III.) If Q is the point required, then, if QP is normal at P, QP = QS, and SP = SG; therefore the triangles QPS and SPG are similar; therefore

$$QP\,.\,PG = SP^2;$$

$$\therefore\ PG\,.\,QG = SP^2 - PG^2 = (x+a)(x-3a),$$

since $SG = x+a$, $PG^2 = 4a\,(x+a)$;

$$\therefore\quad QG^2 = [(x+a)(x-3a)^2]/4a.$$



Again, PG : GQ = GN : GM;

therefore $PG^2 : PG\,.\,GQ = GN : GM$; therefore $GM = \frac{1}{2}(x-3a)$.

Therefore $$QM^2 = [x\,(x-3a)^2]/4a.$$

But $$QM^2 = 4AS\,.\,AM = 4a\,(x+2a+GM);$$

therefore $$[x\,(x-3a)^2]/4a = 4a\,[x+2a+\tfrac{1}{2}\,(x-3a)];$$

therefore $x^3 - 6ax^2 - 15a^2x - 8a^3 = 0$. The only positive root of this is $x = 8a$. Therefore the point P is known; therefore Q is known.

15626. (R. Chartres.)—If C_r, D_r be the combinations of n and $2n$ things respectively taken r together, find the sum of

$$1 + (C_1)^2/D_2 + (C_2)^2/D_4 + (C_3)^2/D_6 + \ldots.$$

Solution by the Proposer.

Legendre's coefficient may be arranged

$$P_n(\cos\theta) = (D_n/2^{2n})\,[\cos n\theta + (C_1^2/D_2)\cos(n-2)\theta + (C_2^2/D_4)\cos(n-4)\theta + \ldots].$$

Let $\theta = 0$; therefore $1 + C_1^2/D_2 + C_2^2/D_4 + \ldots = 2^{2n}/D_n$.

Digitized by Google

15627. (A. H. Bell.)—Give a formula for all integral right-angled triangles where the difference between the sides containing the right angle is 7.

Examples: 5, 12, 13; 8, 15, 17; ....

Solutions (I.) *by* Lt.-Col. Allan Cunningham, R.E., *and* R. W. D. Christie; (II.) *by* J. Blaikie, M.A.; (III.) *by* D. Biddle *and* W. H. Blythe, M.A.; (IV.) *by* C. Bickerdike.

[Solutions far too numerous to allow of publishing more than a few typical ones.]

(I.) The solution may be given in general terms, and then particularized. Let a, b, c be the sides; $c^2 = a^2 + b^2$, $a \sim b = d$ (a given quantity). Then there are *two* general solutions of the Diophantine $c^2 = a^2 + b^2$:—

(i.) $a = \frac{1}{2}(t^2 \sim u^2)$, $b = tu$, $c = \frac{1}{2}(t^2 + u^2)$; t, u both odd, or both even;
(ii.) $a = t^2 \sim u^2$, $b = 2tu$, $c = t^2 + u^2$; t, u one odd, one even;

and the condition $a \sim b = d$ now gives rise to *four* Diophantines in t, u, which may be included in the single form $T^2 - 2u^2 = \pm 2d$ or $\pm d$, where $T = t - u$. But, as it turns out that the $\pm$ signs of $2d$ and of d lead to the *same* solution, one sign (say $-2d$ and $-d$) may be neglected. Next, let T_0, u_0 be the *least* solutions of either of the two Diophantines $T^2 - 2u^2 = +2d$, or $+d$, and let T_1, u_1; T_2, u_2, &c., be the next higher solutions in ascending order; then the general solution (T_n, u_n) is given by $T_n = T_1\tau_n \mp 2u_1v_n$, $u_n = T_1v_n \mp u_1\tau_n$ (both signs $-$, or both $+$), where τ_n, v_n is the nth solution of $\tau^2 - 2v^2 = +1$. These last (τ_n, v_n) may be derived by known rules from the minimum solution $3^2 - 2.2^2 = 1$; or they may be taken direct from the present writer's new *Tables of Quadratic Partitions* (just published), p. 261, $\tau = 3, 17, 99, 577, 3363, \ldots$; $v = 2, 12, 70, 408, 2378, \ldots$.

Ex.—In the present case $d = 7$, and the minimum solutions (T_0, u_0) are

$T_0 = 4$, $u_0 = 1$ in $T^2 - 2u^2 = 14$; $T_0 = 3$, $u_0 = 1$, in $T^2 - 2u^2 = 7$.

Observing that $t = T + u$, and that each pair τ_n, v_n gives *two* values for T_n and u_n (on account of the double sign in the formulæ for T_n, u_n), a few of the values of $T_n, u_n, t_n, a_n, b_n, c_n$ (as far as $n = 2$) may be shown as in table below.

τ	.	3	17	&c.	τ	.	3	17	&c.
v	.	2	12	&c.	v	.	2	12	&c.
t'	4	8, 16	44, 92	&c.	t'	3	5, 13	27, 75	&c.
u	1	5, 11	31, 65	&c.	u	1	3, 9	19, 53	&c.
t	5	13, 27	75, 157	&c.	t	4	8, 22	46, 128	&c.
a	12	72, 304	2332, 10212	&c.	a	15	55, 403	1755, 13575	&c.
b	5	65, 297	2325, 10205	&c.	b	8	48, 396	1748, 13568	&c.
c	13	97, 425	3293, 14437	&c.	c	17	73, 525	2477, 19193	&c.

(II.) We know that, if x and y be integers or multiples of the same quadratic surd, $(x+y)(x-y)$, $2xy$, x^2+y^2 are integers, and may be taken as the lengths of the sides of a right-angled triangle. It is required to find a series u $u_2, u_3, \ldots, u_n$ such that the expression

$$(u_n + u_{n-1})(u_n - u_{n-1}) - 2u_nu_{n-1}$$

Digitized by Google

may be constant in magnitude. If we assume that this difference is alternately positive and negative, we obtain the relation

$$u_n^2 - u_{n-1}^2 - 2u_n u_{n-1} = 2u_{n+1}u_n - u_{n+1}^2 + u_n^2,$$

from which, by reduction, we find $u_{n+1} = 2u_n + u_{n-1}$. Consider now the series (*a*) 1, 2, 5, 12, ...; (*b*) 1, 4, 9, 22, ...; (*c*) 2, 3, 8, 19, ... In series (*a*) we have $(u_n + u_{n-1})(u_n - u_{n-1}) - 2u_n u_{n-1} = \pm 1$. In series (*a*) and (*b*) we have $(u_n + u_{n-1})(u_n - u_{n-1}) - 2u_n u_{n-1} = \pm 7$. Thus series (*a*) gives a set of right-angled triangles 3, 4, 5; 20, 21, 29; ..., in which the two smaller sides differ by unity, and we can obtain triangles fulfilling the conditions of the question by multiplying each side by 7. Series (*b*) and (*c*) give triangles in which the smaller sides differ by 7, viz., (*b*) gives 8, 15, 17; 65, 72, 97; ...; (*c*) gives 5, 12, 13; 55, 48, 73; ....

(III.) By the law enunciated in Question 15360 (*Ed. Times*, June 1, 1903) the integral sides of a right-angled triangle, when reduced to their standard ratio, are as $x^2 + y^2 : x^2 - y^2 : 2xy$. The present Question is satisfied by the two following cases, namely, (1) $2xy \sim (x^2 - y^2) = 7$ and (2) $14xy \sim 7(x^2 - y^2) = 7$. Consequently, we have, from (1),

$$x - y = (2y^2 \pm 7)^{\frac{1}{2}},$$

and, from (2), $x - y = (2y^2 \pm 1)^{\frac{1}{2}}$. In the latter case, the sides of the required triangle are $7(x^2 + y^2)$, $7(x^2 - y^2)$, $14xy$. Taking (1), $2y^2 \pm 7 = \square$ when $y = 1, 2, 3, 4, 8, 9, 19, \ldots$, and the respective sides are 17, 15, 8; 13, 5, 12; 73, 55, 48; 97, 65, 72; 425, 297, 304; 565, 403, 396; 2477, 1755, 1748; .... Taking (2), $2y^2 \pm 1 = \square$ when $y = 1, 2, 5, 12, \ldots$, and the respective sides are 35, 21, 28; 203, 147, 140; 1183, 833, 840; 6895, 4879, 4872; ....

(IV.) Let x and $x + y$ represent the sides of the triangle; then $2x^2 + 14x + 49 = \square$ (b^2, say), which represents the sum of two other squares, which Diophantus, in his Second Book, gives the numbers $2bd/(1 + d^2)$ and $[(1 - d^2)b]/(1 + d^2)$, d being assumed at will. Let $d = 4$; then $\frac{8}{17}b$ and $\frac{15}{17}b$ are sides, the denominator being the hypotenuse. Let $d = 5$; then $\frac{5}{13}b$ and $\frac{12}{13}b$ are sides, the denominator being the hypotenuse. Let $d = \frac{1}{2}$; then $\frac{4}{5}b$ and $\frac{3}{5}b$ are sides, the denominator being the hypotenuse.

P.S.—It is necessary in the present case to make $b = 1$.

Table by A. Thacker (Teacher of Mathematics), copied from a Miscellany, 1743.

Value of d.	Legs of Triangle.	Hypotenuse. $b = 1$.	Value of d.	Legs of Triangle.	Hypotenuse.
2	$\frac{4}{5}b$, $\frac{3}{5}b$	b	9	$\frac{9}{41}b$, $\frac{40}{41}b$	b
4 +	$\frac{8}{17}b$, $\frac{15}{17}b$	b	&c.	&c. &c.	&c.
5 +	$\frac{5}{13}b$, $\frac{12}{13}b$	b	$\frac{1}{3}$	$\frac{3}{10}b$, $\frac{9}{10}b$	b
6	$\frac{12}{37}b$, $\frac{35}{37}b$	b	$\frac{2}{3}$ +	$\frac{4}{5}b$, $\frac{3}{5}b$	b
7	$\frac{7}{25}b$, $\frac{24}{25}b$	b	$\frac{3}{8}$	$\frac{80}{101}$, $\frac{99}{101}$	b
8	$\frac{16}{65}b$, $\frac{63}{65}$	b	$\frac{5}{11}$	$\frac{33}{65}$, $\frac{56}{65}$	b

[N.B.—The + sign shows the solution given, and therefore applicable.]

Digitized by Google

15543. (Professor S. Sircom.)—If D, E, F be any points on the sides BC, CA, AB respectively of a triangle ABC, and X, Y, Z the points in which EF, FD, DE respectively meet the same sides, then

$$(DZ \cdot EX \cdot FY)/(DY \cdot EZ \cdot FX) = -(AF \cdot BD \cdot CE)/(AE \cdot BF \cdot CD).$$

Solutions (I.) *by* A. Holden, B.A.; (II.) *by* C. M. Ross.

(I.) Let ABC be the given triangle, X, Y, Z the points where the transversals EF, FD, DE meet the sides BC, CA, AB respectively. Draw ER parallel to BC, FS parallel to AC, DT parallel to AB. Then

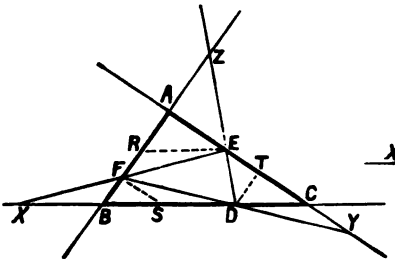


Fig. 1.

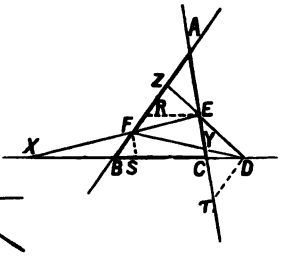


Fig. 2.

$DZ/EZ = BD/RE$, $EX/FX = CE/SF$, and $FY/DY = AF/TD$, from similar triangles; therefore we have

$$(DZ \cdot EX \cdot FY)/(EZ \cdot FX \cdot DY) = (BD \cdot CE \cdot AF)/(RE \cdot SF \cdot TD) \ldots (1).$$

Again the triangles ARE and ABC are similar; therefore

$$RE/EA = BC/CA.$$

Similarly $SF/FB = CA/AB$ and $TD/DC = AB/BC$; therefore

$$(RE \cdot SF \cdot TD)/(EA \cdot FB \cdot DC) = 1;$$

therefore $RE \cdot SF \cdot TD = EA \cdot FB \cdot DC = -AE \cdot BF \cdot CD$.

Therefore, substituting in (1), we have

$$(DZ \cdot EX \cdot FY)/(EZ \cdot FX \cdot DY) = -(BD \cdot CE \cdot AF)/(AE \cdot BF \cdot CD).$$

The same proof holds for any other positions of D, E, F as in Fig. 2.

(II.) From the triangle AEF and transversal CX,

$$CE \cdot AB \cdot FX = CA \cdot BF \cdot XE \ldots (1);$$

from the triangle BFD and transversal AY,

$$FA \cdot BC \cdot DY = AB \cdot CD \cdot YF \ldots (2);$$

and, from the triangle CDE and transversal BZ,

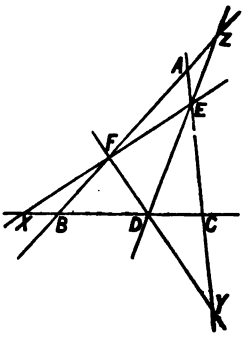
$$DB \cdot CA \cdot EZ = BC \cdot AE \cdot ZD \ldots (3).$$

Multiplying (1), (2), and (3) together,

$$(XE \cdot YF \cdot ZD)(AE \cdot BF \cdot CD) = (DY \cdot EZ \cdot FX)(CE \cdot DB \cdot FA);$$

therefore

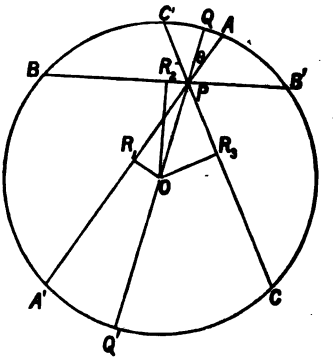
$$(DZ \cdot EX \cdot FY)/(DY \cdot EZ \cdot FX) = -(AF \cdot BD \cdot CE)/(AE \cdot BF \cdot CD).$$



Digitized by Google

15488. (G. Heppel, M.A.)—P is any point within a circle. APA′, BPB′, CPC′ are three chords making angles of 120° with one another. Prove that AP + BP + CP = A′P + B′P + C′P, and that each ranges between 3 and $\sqrt{3}$ times the radius.

Solution by Constance I. Marks, B.A.



The construction in the figure is self-evident. Also

$$\angle APC' = 60°,$$

and similarly for each angle similarly formed. Let θ be the angle which is not greater than $60° - \theta$. Then

$$A'P - AP = 2OP\cos\theta,$$

$$B'P - BP = -2OP\cos(60° + \theta)$$
$$= -OP\cos\theta + OP\sqrt{3}\sin\theta,$$

$$C'P - CP = -2OP\cos(60° - \theta)$$
$$= -OP\cos\theta - OP\sqrt{3}\sin\theta.$$

By addition, $(A'P + B'P + C'P) - (AP + BP + CP) = 0,$

or $$AP + BP + CP = A'P + B'P + C'P.$$

Again, if P coincides with O,

$$AA' + BB' + CC' = 2(AP + BP + CP) = 3 \times \text{diameter};$$

$\therefore$ AP + BP + CP = 3r = A′P + B′P + C′P = a maximum (r = radius).

Finally, projecting the diameter QQ′ orthogonally on each of the given chords

$$AA' + BB' + CC' > 2r\cos\theta + 2r\cos(60° + \theta) + 2r\cos(60° - \theta) > 4r\cos\theta.$$

Making $\theta = 30°$ = a maximum, and $\cos\theta$ therefore a minimum,

$$AA' + BB' + CC' > 4r \tfrac{1}{2}\sqrt{3} > 2r\sqrt{3},$$

whence AP + BP + CP or A′P + B′P + C′P is always greater than $r\sqrt{3}$.

15594. (J. J. Barniville, B.A.)—Having

$$u_1 = 2b^2 - a^2, \qquad u_2 = a^2 + 4ab + 2b^2,$$
$$u_3 = a^2 + 8ab + 14b^2, \qquad u_4 = 7a^2 + 36ab + 46b^2, \quad \ldots,$$

and $v_1 = a^2 + 2ab + 2b^2$, $v_2 = a^2 + 6ab + 10b^2$, $v_3 = 5a^2 + 26ab + 34b^2$, ..., the scale of each series being $x^3 - 3x^2 - 3x + 1$, prove that $u_n^2 + u_{n+1}^2 = 2v_n^2$.

Solution by C. M. Ross.

The scale of relation of the series

$$2b^2 - a^2 + x(a^2 + 4ab + 2b^2) + (a^2 + 8ab + 14b^2)x^2 + \ldots$$

is $1 - 3x - 3x^2 + x^3$, and the generating function is

$$[2b^2 - a^2 + 4x(a^2 + ab - b^2) + (a^2 - 4ab + 2b^2)x^2] / (1 - 3x - 3x^2 + x^3).$$

Digitized by Google

This expression is equivalent to the partial fractions

$$-\tfrac{1}{3}(a^2+2ab-2b^2)\frac{1}{1+x}$$
$$-\frac{a^2(7-4\sqrt{3})+2ab(-5+3\sqrt{3})+2b^2(2-\sqrt{3})}{3(\sqrt{3}-1)}\frac{1}{x-2+\sqrt{3}}$$
$$+\frac{a^2(7+4\sqrt{3})+2ab(-5-3\sqrt{3})+2b^2(2+\sqrt{3})}{3(\sqrt{3}+1)}\frac{1}{x-2-\sqrt{3}}.$$

The sum of the coefficients of x^{n-1} and x^n in the several terms of this expression are equal to u_n and u_{n+1}. Similarly, the sum of the coefficients of x^{n-1} in the expression

$$\tfrac{1}{3}(a^2+2ab-2b^2)\frac{1}{1+x}$$
$$-\frac{a^2(-5+3\sqrt{3})+4ab(2-\sqrt{3})+2b^2(-1+\sqrt{3})}{3(\sqrt{3}-1)}\frac{1}{x-2+\sqrt{3}}$$
$$+\frac{a^2(-5-3\sqrt{3})+4ab(2+\sqrt{3})+2b^2(-1-\sqrt{3})}{3(\sqrt{3}+1)}\frac{1}{x-2-\sqrt{3}}.$$

is equal to v_n.

Hence, grouping together like terms in the squares of the coefficients of x^{n-1} and x^n thus found, it follows that $u_n^2+u_{n-1}^2=2v_n^2$.

15606. (Rev. J. Cullen.)—Given four vectors α, β, γ, δ, show that, if $\phi(t)=\mathrm{V}(\alpha+t\beta)(\gamma+t\delta)^{-1}$, then $\mathrm{U}\Sigma\mathrm{V}\,\phi(t_1)\phi(t_2)$ is invariable for all values of t_1, t_2, t_3, provided $t_1>t_2>t_3$. Hence obtain the equation of the plane formed by the intersection of the reciprocal cone (whose vertex is the origin) with the cylindroid determined by two given screws $(\sigma,\ \omega)$ and $(\sigma_1,\ \omega_1)$.

Solution by the Proposer.

Put $(\gamma+t_1\delta)^2=\mathrm{T}_1^2$, ...; then

$$\phi(t_1)=\mathrm{T}_1^{-2}\mathrm{V}(\alpha+t_1\beta)(\gamma+t_1\delta)=\mathrm{T}_1^{-2}(\xi+t_1\eta+t_1^2\zeta)\ \text{(suppose)}.$$

Hence $$\Sigma\mathrm{V}\,\phi(t_2)\phi(t_3)=(t_1-t_2)(t_2-t_3)(t_3-t_1)\,\mathrm{T}_1^{-2}\mathrm{T}_2^{-2}\mathrm{T}_3^{-2}\,\mathrm{K},$$

where $$\mathrm{K}=\gamma^2.\mathrm{V}\eta\zeta+2\mathrm{S}\gamma\delta.\mathrm{V}\zeta\xi+\delta^2.\mathrm{V}\xi\eta.$$

Therefore $\mathrm{U}\Sigma\mathrm{V}\,\phi(t_2)\phi(t_3)=\mathrm{U}\kappa$, which is independent of t_1, t_2, t_3 provided $t_1>t_2>t_3$.

The equation of the plane through $\phi(t_1)$, $\phi(t_2)$, $\phi(t)_3$ is

$$\mathrm{SK}[\rho-\phi(t_1)]=0$$

or $$\mathrm{SK}\rho-\mathrm{S}\xi\eta\zeta=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (1).$$

Now, since the vector equation of the cylindroid is

$$\rho=\mathrm{V}(\sigma+t\sigma_1)(\omega+t\omega_1)^{-1}+x(\omega+t\omega_1),$$

we obtain on putting $x=0$ the perpendiculars on the generators which lie on the cone $\mathrm{S}\sigma\rho\mathrm{S}\omega_1\rho-\mathrm{S}\sigma_1\rho\,\mathrm{S}\omega\rho=0$, and from the above we need only to replace α, β, γ, δ by σ, σ_1, ω, ω_1 in K, ξ, η, ζ in (1) to obtain the equation of the plane.

Digitized by Google

Algebraical Note.

By A. M. Nesbitt, M.A.

To find the number of ways in which it is possible to place n letters each into a wrong envelope.

The following method of working this familiar question may be found interesting.

Take as a first approximation the total number of ways, $n!$, in which the n letters can be placed in their envelopes. This is too large, and therefore from it we subtract the number in which a is placed right and the rest anyhow, the number in which b is placed right and the rest anyhow, ...; thus subtracting in all $n(n-1)!$. But now we have taken off too large a number, and must therefore add the number of ways in which a, b are put right and the rest anyhow, and so for each and every pair: this gives the term $+\frac{n(n-1)}{1.2}(n-2)!$. Here again we have added too large a number and must compensate by taking off ${}_nC_3(n-3)!$; this in turn requires the addition of ${}_nC_4(n-4)!$; and so on. Hence the required number is

$$ {}_nC_0 n! - {}_nC_1(n-1)! + {}_nC_2(n-2)! - {}_nC_3(n-3)! + \ldots + (-1)^n {}_nC_n 0!,$$

i.e.,
$$n![1 - 1/1! + 1/2! - 1/3! + \ldots + (-1)^n 1/n!].$$

Cor.—If there are more envelopes than letters—*i.e.*, suppose there to be $m-n$ blank envelopes—we are at once led by similar reasoning to the fact that the number of ways of putting the letters in envelopes so that none shall go right is

$$ {}_nC_0\,{}_mP_n - {}_nC_1\,{}_mP_{n-1} + {}_nC_2\,{}_mP_{n-2} - \ldots + (-1)^{n-1}\,{}_nC_{n-1}\,{}_mP_1 + (-1)^n\,{}_nC_n,$$

i.e.,
$$\sum_{r=0}^{r=n} (-1)^r \frac{m!}{(m-n+r)!} \frac{n!}{r!(n-r)!}$$

or
$$\sum_{r=0}^{r=n} (-1)^r m!\, n!/[r!(n-r)!(m-n+r)!];$$

so that the chance that all the letters go astray is got by substituting $(m-n)!$ for $m!$ in the above expression.

The case in which there are more letters than envelopes presents no special difficulty.

15198. (Professor Sanjána, M.A.)—ACA′, BCB′ are two fixed conjugate diameters of a hyperbola, and CP, CD are two conjugate semi-diameters: if AP, BD meet in O, and A′P, B′D in Q, prove that PODQ is a parallelogram, whose diagonal OQ is of fixed length and lies along an asymptote. Find also the maximum area of this parallelogram.

Solution by A. M. Nesbitt, M.A.

Taking the conjugate diameters AA′, BB′ for axes and (x', y') for coordinates of P, we get the following equations:—

$$y/(x-a) = y'/(x'-a) \text{ for AP},$$
$$(y+b)/x = (bx'/a+b)/(ay'/b) = y'/(x'-a) \text{ for B'D},$$
$$(y-b)/x = y'/(x'+a) \text{ for BD}.$$

Digitized by Google

Hence B'D is parallel to AP; so that OQ is a parallelogram. Also, since $y'^2/(x'^2-a^2) = b^2/a^2$, we may take $y/(x-a) = bm/a$ for AP and

$$(y-b)/x = b/am \text{ for BD,}$$

and, on multiplying up and adding, we find $(m+1)(ay-bx) = 0$; so that AP and BD intersect on the asymptote $ay = bx$. [If $m = -1$, the two lines coincide with AB and may therefore be said to intersect on the other asymptote.] Hence we get, for co-ordinates of O, $[am/(m-1), bm/(m-1)]$. Now we may similarly take

$$y/(x+a) = b/am, \quad (y+b)/x = bm/a$$

for equations to A'P and BD, and these meet in $[a/(m-1), b/(m-1)]$, which are co-ordinates of Q. The difference of the abscissæ of O and Q is thus a, and this is the (oblique) projection of OQ on the x axis. The area of the parallelogram is thus $ab \sin \omega$; so that its area is constant.

12229. (Professor Lévy.)—Un régiment marche à la cadence de 120 pas par minute; on demande quels sont les soldats qui poseront le pied droit sur le sol en même temps que la musique. Même problème en changeant la cadence.

Solution by W. J. Greenstreet, M.A.

Sound travels at the rate of 1120 ft. per second. The troops are marking time, or walking in a straight line, at 120 paces a minute, *i.e.*, at 2 paces a second, *i.e.*, each man places his right foot on the ground once in a second and his left foot half a second after. Then men at intervals of $1120k$ ft., where k is integral, will appear to the spectator to be keeping time with the band. If the time is changed, the 2 paces a second becomes 2 paces in x seconds. In x seconds sound travels $1120x$ ft. Hence the intervals above must be $1120xk$ ft.

12906. (Professor Orchard, B.Sc., M.A.)—Show from first principles that the time of a small oscillation of a hemisphere on any plane must be proportional to the square root of the radius.

Solution by A. M. Nesbitt, M.A.

The time of an oscillation can only depend on the mass and radius of the hemisphere and on g. If we take it to vary as $m^\alpha r^\beta g^\gamma$, its dimensions in mass, length, and time are respectively α, $\beta+\gamma$, and -2γ. Hence $\alpha = \beta+\gamma = 0$ and $\gamma = -\frac{1}{2}$; thus the time of an oscillation will be proportional to $\sqrt{(r/g)}$.

Digitized by Google

15476. (A. M. Nesbitt, M.A.) — Find the linear factors of $abcfgh)(xyz)^2$ under the condition $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$. Apply the method to the case $(b-c)(x^2+2yz)+(c-a)(y^2+2zx)+(a-b)(z^2+2xy)=0$.

Solutions (I.) *by the* Proposer; (II.) *by* Professor Sanjána, M.A.

(I.) If we write $x=\alpha+\alpha'$, $y=\beta+\beta'$, choosing α', β' so that $a\alpha'+h\beta'+gz=0$, $h\alpha'+b\beta'+fz=0$, the expression under the given conditions reduces to $a\alpha^2+b\beta^2+2h\alpha\beta$; and the problem is at once solved.

In the particular case quoted we have only to put $\alpha'=\beta'=z$ and the expression reduces to $(b-c)\alpha^2+(c-a)\beta^2+2(a-b)\alpha\beta=0$.

Referring to Question 15290, wherein a solution is asked for of the three simultaneous equations $x^2+2yz=a$, $y^2+2zx=b$, $z^2+2xy=c$, it will be seen at once that

$$(b-c)(x^2+2yz)+(c-a)(y^2+2zx)+(a-b)(z^2+2xy)=0,$$

while $$(x-y)(x+y-2z)=a-b.$$

But $$x-y=\alpha-\beta \quad \text{and} \quad x+y-2z=(x+y+z)-3z;$$

and $(x+y+z)^2=a+b+c=d^2$ by hypothesis; so that, if $3z=d-\alpha-\beta$, we shall have

$$\alpha^2-\beta^2=a-b \quad \text{and} \quad (b-c)\alpha^2+(c-a)\beta^2+2(a-b)\alpha\beta=0,$$

x and y being respectively $z+\alpha$ and $z+\beta$.

The case of the simultaneous equations in Question 15290 is only worth quoting as an instance of the way in which the solution of a purely algebraical question may at times be facilitated by a knowledge of analytical geometry.

(II.) Solving for x the equation $f(xy)=0$, we get

$$ax+hy+gz=\pm\sqrt{[(h^2-ab)y^2+2(hg-af)yz+(g^2-ac)z^2]};$$

the right is linear when $(hg-af)^2=(h^2-ab)(g^2-ac)$, *i.e.*, when the given condition is satisfied. Thus the factors are

$$[ax+hy+gz+\sqrt{(h^2-ab)}\,y+\sqrt{(g^2-ac)}\,z]$$

and

$$[ax+hy+gz-\sqrt{(h^2-ab)}\,y-\sqrt{(g^2-ac)}\,z],$$

with the constant multiplier $1/a$. In the case proposed the condition is satisfied, as $\begin{vmatrix} b-c & a-b & c-a \\ a-b & c-a & b-c \\ c-a & b-c & a-b \end{vmatrix}$ is evidently zero. Also

$$h^2-ab=g^2-ac=-(hg-af)=a^2+b^2+c^2-ab-bc-ca\equiv e^2,$$

suppose. Then the factors are $1/(b-c)$ and

$$[(b-c)z+(a-b+e)y+(c-a-e)z], \quad [(b-c)x+(a-b-e)y+(c-a+e)z]$$

These can also be exhibited in two other forms by solving for y and z instead of x.

Digitized by Google

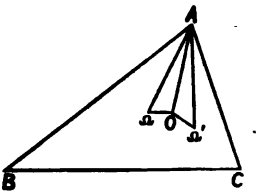
9870. (R. TUCKER, M.A.)—O, Ω, Ω' are the in-centre and Brocard points of a triangle ABC. Ω_1, Ω_1'; Ω_2, Ω_2'; Ω_3, Ω_3' are the images of Ω, Ω' with respect to AO, BO, CO. Prove that, if ρ_1, ρ_2 are the radii of the circles round Ω, Ω_1, Ω_2, Ω_3; Ω', Ω_1', Ω_2', Ω_3' respectively, and if $\lambda^2 \equiv a^2b^2 + b^2c^2 + c^2a^2$, then

$$\rho_1^2 \sim \rho_2^2 = [abc\,\Pi(a-b)]/\lambda^2, \quad \rho_1^2 + \rho_2^2 = abc\,[\Sigma a\,\Sigma a^3 - (\Sigma a^2)^2]/(2s\lambda^2).$$

The equations to the two circles in trilinear co-ordinates are easily obtained from the fact that they are concentric with the in-circle of ABC.

Solution by Professor SANJÂNA, M.A.

It is easily seen that the circle $\Omega\Omega_1\Omega_2\Omega_3$ has O for centre and OΩ for radius; so for $\Omega'\Omega_1'\Omega_2'\Omega_3'$. Thus we have to find the distances of the in-centre from the Brocard points. This may be done by the co-ordinate method; or, more elementarily, as below.



We have
$$\Omega A = 2R\sin\omega\,(b/a) = 4R\Delta b/\lambda a = b^2c/\lambda;$$
$$OA = 4R\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C = \sqrt{[bc\,(s-a)/s]},$$
$$\Omega' A = 2R\sin\omega\,(c/a) = 4R\Delta c/\lambda a = bc^2/\lambda;$$

also
$$\angle\,\Omega AO = \angle\,\Omega' AO = \tfrac{1}{2}A - \omega.$$

Hence
$$\rho_1^2 - \rho_2^2 = O\Omega^2 - O\Omega'^2 = \Omega A^2 + OA^2 - 2\Omega A\,.\,OA\cos(\tfrac{1}{2}A-\omega) - (\Omega' A^2 + OA^2 - 2\Omega' A\,.\,OA)\cos(\tfrac{1}{2}A-\omega)$$
$$= \Omega A^2 - \Omega' A^2 - 2OA\,(\Omega A - \Omega' A)\cos(\tfrac{1}{2}A-\omega)$$
$$= (\Omega A - \Omega' A)\,[\Omega A + \Omega' A - 2OA\cos(\tfrac{1}{2}A-\omega)]$$

Now $\cos(\tfrac{1}{2}A-\omega) = \dfrac{a^2+bc}{\lambda}\cos\tfrac{1}{2}A$, and $r\cot\tfrac{1}{2}A = s-a$;

therefore
$$\rho_1^2 - \rho_2^2 = \frac{bc\,(b-c)}{\lambda^2}\,[bc\,(b+c) - 2OA\,(a^2+bc)\cos\tfrac{1}{2}A]$$
$$= \frac{bc\,(b-c)}{\lambda^2}\,[bc\,(b+c) - 2\,(s-a)\,(a^2+bc)]$$
$$= \frac{bc\,(b-c)}{\lambda^2}\,[(b+c)(bc-a^2-bc) + a\,(a^2+bc)]$$
$$= \frac{bc\,(b-c)}{\lambda^2}\,[a^2\,(a-b-c) + abc]$$
$$= \ldots.$$

The value of $\rho_1^2 + \rho_2^2$ may be similarly found.

Digitized by Google

8059. (T. MUIR, LL.D., F.R.S.E.)—Prove that the Pfaffian

$$\begin{vmatrix} a_1+a_2, & a_1+a_3, & a_1+a_4, & \ldots, & a_1+a_{2n} \\ & a_2+a_3, & a_2+a_4, & \ldots, & a_2+a_{2n} \\ & & a_3+a_4, & \ldots, & a_3+a_{2n} \\ & & \ldots & \ldots & \ldots \\ & & & & a_{2n-1}+a_{2n} \end{vmatrix} = 2^{n-1}(a_1a_3a_5\ldots a_{2n-1} + a_2a_4a_6\ldots a_{2n}).$$

Solution by Professor NANSON.

Let P denote the Pfaffian and Δ the generating zero-axial, so that $\Delta = P^2$. Then P is of the first degree in each frame line and therefore of the form $\lambda a_1a_2 + \mu a_1 + \nu a_2 + \rho$, where λ, μ, ν, ρ do not contain a_1, a_2. Next, Δ contains no term with $a_1^2a_2^2$ as factor and has two rows identical if $a_1 = a_2 = 0$. Hence $\lambda = 0$, $\rho = 0$. Further, Δ is unaltered by a cyclic interchange of the suffixes 1, 2, ..., $2n$. Hence $\mu = \nu$ and P is of the form $k(a_1a_3\ldots a_{2n-1} + a_2a_4\ldots a_{2n})$, where k is numerical. Now, making $a_1 = a_2 = \ldots = 1$, the terms of P are each numerically 2^n, and of these terms there is one more positive than negative. Hence $k = 2^{n-1}$.

15629. (A. W. T.)—A number is raised to the power (-1) and reduced to a decimal. It is found that the original number is just n times the digit nearest the decimal point. Find the limits between which the number lies.

Solution by Lt.-Col. ALLAN CUNNINGHAM, R.E.

Let $a_1, a_2, \ldots$ be the digits of 1/N, expressed as a decimal, and let a_r be the first digit greater than 0 ($a_1 = a_2 = \ldots = a_{r-1} = 0$). Then $N > 10^{r-1}$, but $< 10^r$, and $N = n \,.\, a_r$. Also $1/N = a_r/10^r + a_{r+1}/10^{r+1} + \ldots$. Hence (except when a_r is the only finite digit)

$$1/N < a_r/10^r \quad \text{and} \quad 1/N < (n \,.\, a_r)/(n \,.\, 10^r) = N/(n \,.\, 10^r),$$

whence $N > \sqrt{(n \,.\, 10^r)}$. Hence $N > \sqrt{(n \,.\, 10^r)}$, but $< 10^r$.

Also $N^2 > n \,.\, 10^r$, whence $N > n\,(10^r/N)$. Hence $N > n$ (because $10^r/N > 1$). This lower limit of N is not so high as the previous one [$N > \sqrt{(n \,.\, 10^r)}$], but has the advantage of not involving r.

15527 (with changed notation). (L. ISSERLIS.)—Two conics touch at O. The distances of their centres from the tangent at O are h, H and their radii of curvature at O are r, R. The diameter of the first through O meets it again in A and the tangent at A meets the second conic in P, Q. Prove that, if $1/h - 1/H = 1/\sqrt{(rh)}$, then the area of the triangle OPQ is $4R^{\frac{1}{2}}r^{-\frac{1}{2}}h^{\frac{1}{2}}$.

Digitized by Google

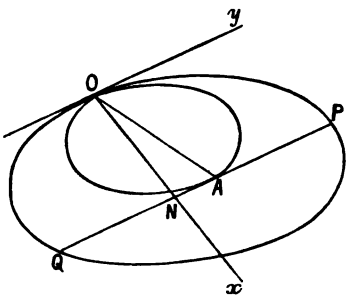
Solution by R. F. Davis, M.A.

If (h, k) are the co-ordinates of the centre of the first conic referred to the normal at O as axis of x and the tangent at O as the axis at y, the equation to the conic is

$$(k^2 + hr)\,x^2 - 2hkxy + h^2(y^2 - 2rx) = 0.$$

Similarly, the equation to the second conic is

$$(K^2 + HR)\,x^2 - 2HKxy + H^2(y^2 - 2Rx) = 0.$$



The co-ordinates of A are of course $(2h, 2k)$, and the equation to the tangent at A is $x = 2h$.

To determine P, Q we have

$$4h^2(K^2 + HR) - 4HKhy + H^2(y^2 - 4Rh) = 0,$$

$$y^2 - \frac{4Kh}{H}y + \frac{4h^2(K^2 + HR) - 4H^2hR}{H^2} = 0;$$

therefore $(PQ)^2 = (y_1 \sim y_2)^2 = (y_1 + y_2)^2 - 4y_1y_2$

$$= \frac{16K^2h^2}{H^2} - \frac{16h^2(K^2 + HR) - 16H^2hR}{H^2}$$

$$= 16hR - 16h^2R/H = 16Rh(1 - h/H).$$

$$4(\text{area } \Delta OPQ)^2 = PQ^2 \cdot ON^2 = PQ^2 \cdot 4h^2;$$

therefore $$(\text{area } \Delta OPQ)^2 = 16Rh^3(1 - h/H).$$

This result does not involve r; which is not to be wondered at.

Note also the expression for the area vanishes when $h = H$. In this case the line $x = 2h$ is a tangent to both curves.

If now the condition $1/h - 1/H = 1/\sqrt{(rh)}$ be introduced, the above expression for Δ takes the form given in the Question.

15471. (Rev. T. Roach, M.A.)—Prove

(1) $\cosh^{-1}(\sec x) = \log\cot(\frac{1}{4}\pi - \frac{1}{2}x)$ or $\log\tan(\frac{1}{4}\pi - \frac{1}{2}x)$;

(2) $\sinh^{-1}(\tan x) = \log\cot(\frac{1}{4}\pi - \frac{1}{2}x)$ or $\log\tan(\frac{1}{4}\pi + \frac{1}{2}x)$.

Solutions (I.) *by* W. Scrimgeour, M.A., B.Sc., *and others*; (II.) *by* W. R. Bruce.

(I.) (1) Let $\cosh^{-1}(\sec x) = y$; therefore $\cosh y = \sec x$; therefore $\frac{1}{2}(e^y + e^{-y}) = \sec x$. Solving for e^y, after reducing,

$$e^y = \sec x \pm \sqrt{(\sec^2 x - 1)};$$

therefore $$e^y = \sec x + \tan x \quad \text{or} \quad e^y = \sec x - \tan x;$$

therefore

$$e^y = (1 + \tan\tfrac{1}{2}x)/(1 - \tan\tfrac{1}{2}x) \quad \text{or} \quad e^y = (1 - \tan\tfrac{1}{2}x)/(1 + \tan\tfrac{1}{2}x);$$

Digitized by Google

therefore $y = \log\tan(\frac{1}{4}\pi+\frac{1}{2}x)$ or $y = \log\tan(\frac{1}{4}\pi-\frac{1}{2}x)$;

therefore $\cosh^{-1}\sec x = \log\cot(\frac{1}{4}\pi-\frac{1}{2}x)$ or $\log\tan(\frac{1}{4}\pi-\frac{1}{2}x)$.

(2) Similarly, let $\sinh^{-1}(\tan x) = z$; therefore $\sinh z = \tan x$; therefore $\frac{1}{2}(e^z - e^{-z}) = \tan x$. Therefore, solving for e^z, as before,

$$e^z = \tan x \pm \sqrt{(\tan^2 x + 1)};$$

therefore $e^z = \tan x + \sec x$ or $e^z = \tan x - \sec x$

therefore $e^z = (1+\tan\frac{1}{2}x)/(1-\tan\frac{1}{2}x)$

or $e^z = [(1-\sin x)/-\cos x] = (\tan\frac{1}{2}x-1)/(\tan\frac{1}{2}x+1)$;

therefore $z = \log\tan(\frac{1}{4}\pi+\frac{1}{2}x)$ or $z = \log\tan(\frac{1}{2}x-\frac{1}{4}\pi)$.

The latter is negative when $x < \frac{1}{2}\pi$; therefore

$$\sinh^{-1}(\tan x) = \log\cot(\frac{1}{4}\pi-\frac{1}{2}x) \quad \text{or} \quad = \log\tan(\frac{1}{4}\pi+\frac{1}{2}x).$$

Note.—The common value of (1) and (2) is $\log\cot(\frac{1}{4}\pi-\frac{1}{2}x)$, the inverse Gudermannian of x. Thus

$$\cosh^{-1}(\sec x) = \sinh^{-1}(\tan x) = \log\cot(\tfrac{1}{4}\pi-\tfrac{1}{2}x) = gd^{-1}x$$

when only the positive value of the above radicals is taken, as in finding, say, $\cosh^{-1}y$.

(II.) We know that

$\cosh^{-1}(\sec x) = \log[\sec x + \sqrt{(\sec^2 x - 1)}] = \log(\sec x + \tan x) = y$ (say).

Differentiating, we have

$$dy = [(\sec x\tan x + \sec^2 x)/(\sec x + \tan x)]\,dx = \sec x\,dx = dx/\cos x.$$

Therefore $y = \int dx/\cos x = \log\cot(\frac{1}{4}\pi-\frac{1}{2}x)$,

which is a well-known integral.

The solution $\log\tan(\frac{1}{4}\pi+\frac{1}{2}x)$ follows at once.

To show that $\sinh^{-1}(\tan x) = \log\cot(\frac{1}{4}\pi-\frac{1}{2}x)$, we have

$\sinh^{-1}(\tan x) = \log[\tan x + \sqrt{(\tan^2 x + 1)}] = \log(\tan x + \sec x) = y$ (say).

Therefore

$$dy = \sec x\,[(\sec x + \tan x)/(\sec x + \tan x)]\,dx = \sec x\,dx = dx/\cos x.$$

Therefore $y = \int dx/\cos x = \log\cot(\frac{1}{4}\pi-\frac{1}{2}x)$ or $\log\tan(\frac{1}{4}\pi+\frac{1}{2}x)$.

9308. (Professor NILKANTHA SARKAR, M.A.)—A uniform rod rests in a horizontal position, being supported by two smooth pegs equidistant from its centre. If the distance between the pegs $= 1/\sqrt{3}$ times the length of the rod, prove that, if one of the pegs be suddenly removed, the pressure on the other will be initially unaltered.

Solution by FRANCES E. CAVE.

Let θ be the inclination of the rod to the horizon at any instant before it leaves the second peg, r the distance of its centre from the peg, P the pressure. With the usual notation,

$$mk^2\ddot{\theta} = Pr, \quad (m/r)(d/dt)(r^2\dot{\theta}) = mg\cos\theta - P;$$

therefore initially $Pr^2 = k^2(mg - P)$; therefore $P = [k^2/(k^2+r^2)]\,mg = \frac{1}{2}mg$, since $r^2 = k^2$.

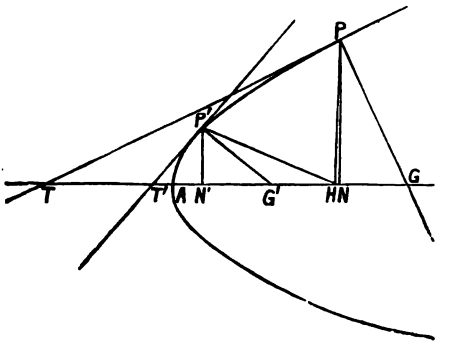
Digitized by Google

15638. (A. M. NESBITT, M.A.)—The normals to a parabola at P and P′ meet the axis in G and G′. If H be the mid-point of GG′, prove geometrically that HP = HP′.

Solutions (I.) *by* R. TUCKER, M.A., C. BICKERDIKE, *and many others;*
(II.) *by* KELA DEVA RAU, B.A.

(I.) GG′ = NN′ and

$$HP^2 \sim HP'^2 = 4AS.AN + 4AS^2 - 4AS.HG + HG^2$$
$$\sim (4AS.AN' + 4AS^2 + 4AS.HG' + HG'^2)$$
$$= 4AS.NN' - 4AS.GG' = 0, \ldots.$$

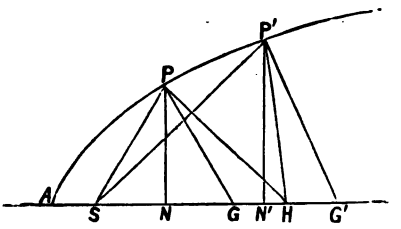


(II.) Join SP and SP′. From P and P′ draw PN and P′N′ perpendicular to the axis.

$$PH^2 = SP^2 + SH^2 - 2SH.SN$$
$$= SG^2 + SH^2 - 2SH.SN.$$

Similarly,

$$P'H^2 = SG'^2 + SH^2 - 2SH.SN';$$



therefore $P'H^2 - PH^2 = SG'^2 - SG^2 - 2SH(SN' - SN)$
$= (SG' + SG)(SG' - SG) - 2SH.NN$
$= 2SH.GG' - 2SH.NN'$
$= 2SH(GG' - NN') = 0.$

Since GG′ = NN′ + N′G′ − NG = NN′; therefore P′H = PH.

15371. (Professor NEUBERG.)—Trouver le lieu des centres des hyperboles équilatères qui passent par deux points donnés A, B et dont une asymptote passe par un troisième point donné C.

Digitized by Google

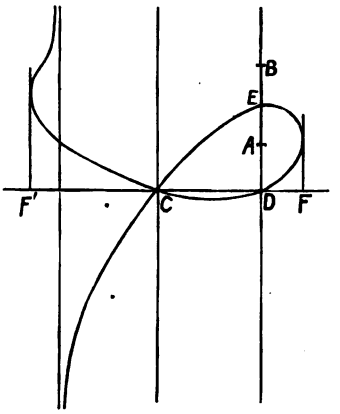
Solutions (I.) *by* A. M. NESBITT, M.A.; (II.) *by* Professor SANJÁNA, M.A.

Take C for origin, and let the co-ordinates of A, B be (c, a), (c, b). Let the equation to the asymptotes be $y = mx$, $my + x = h$. Then the equation to the curve will be

$$(y - mx)(my + x - h) = k^2,$$

and we have at once

$$(a - mc)(ma + c - h) = (b - mc)(mb + c - h).$$



But, if x, y be the co-ordinates of the centre (*i.e.*, of the intersection of the asymptotes), we have

$$m = y/x,$$

$$h = x(1 + m^2) = (x^2 + y^2)/x;$$

whence the equation to the locus is

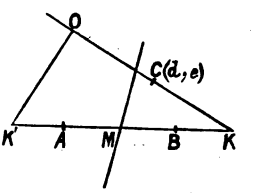
$$x(x^2 + y^2) = xy(a + b) + c(x^2 - y^2),$$

a circular cubic with $x = -c$ as its real asymptote and a real double point at C. The locus will be found to cut the x axis at D, and, if CF = CF' = CE, where E is the middle point of AB, then CF and CF' are the limits of x for *real* values of y.

Moreover the curve passes through E, the values of dy/dx at D and E being respectively $c/(a + b)$ and $-\{c^2 - [\frac{1}{2}(a + b)]^2\}/[c - (a + b)]$; and at the double point the two values are of opposite sign. The curve cuts the asymptote at the point whose ordinate is $2c^2/(a + b)$; hence it is approximately as depicted.

(II.) Take the mid-point M of AB as origin, AB as axis of x, C to be (d, e); MK = MK' = k. Join KC, and draw K'O perpendicular; then O is one position of the centre. The line KO is

$$y = (x - k)e/(d - k),$$



the line K'O is

$$y = -(x + k)(d - k)/e.$$

Multiplying, $y^2 + x^2 = k^2 = (yd - ex)^2/(y - e)^2$ from the first; hence

$$y^2(y - e)^2 + x^2y^2 - 2x^2ye = y^2d^2 - 2yxde,$$

i.e.,

$$y(y - e)^2 + yx^2 = 2x^2e + yd^2 - 2xde,$$

which is a cubic, having $y - e = 0$ as one asymptote.

Remarks by the PROPOSER.—1. D'après la solution de M. Sanjána, si l'on prend sur AB les distances MK = MK', la projection O de K' sur la droite CK et la projection O' de K sur la droite CK' sont deux points du lieu cherché.

Ces points O, O' sont sur la circonférence de centre M et de rayon MK.

2. Si le point K est à l'infini sur AB, la droite CK devient parallèle à AB et la droite K'O se transporte à l'infini. Donc la courbe passe par le point à l'infini de AB.

Digitized by Google

3. Soient ω, ω' les points situés à l'infini sur toutes les circonférences du plan ABC. Si l'on prend pour CK la droite imaginaire $C\omega$, la droite K'O passera également par ω; car deux droites passant par ω ont des équations de la forme $y = x\sqrt{-1} + p$, $y = x\sqrt{-1} + q$, et elles vérifient la condition de perpendicularité. Ainsi, le point ω appartient au lieu cherché, et il en est de même du point ω'. Nous avons ainsi sur la droite de l'infini trois points du lieu; ce lieu est donc une cubique circulaire.

4. Si l'on suppose MK = MK' = MC, le point O tombe en C; la cubique passe donc en C. C est même un point double, ayant pour tangentes les droites correspondantes CK, CK'; c'est ce qu'on voit en considérant les points du lieu situés sur les droites menées par C et infiniment peu différentes de CK, CK'.

5. Trois des hyperboles équilatères se réduisent à des couples de droites rectangulaires. Ces couples sont: (1) la droite AB et la perpendiculaire abaissée de C sur AB (d'où le point D trouvé par M. Nesbitt); (2) la droite CB et la perpendiculaire abaissée de A sur CB (ce qui correspond à K placé en B); (3) la droite AC et la perpendiculaire abaissée de B sur AC. Si on place K en M, O coincide avec M.

6. Les points K, K' décrivent des ponctuelles semblables, qui sont projectives avec le faisceau des droites CK; donc la droite KO perpendiculaire à CK enveloppe une parabole.

15438. (G. Heppel, M.A.)—P is any point within a circle. APA', BPB', CPC' are three chords making angles of 120° with one another. Prove that AP + BP + CP = A'P + B'P + C'P, and that each ranges between 3 and $\sqrt{3}$ times the radius.

Solution by the Proposer.

O is the centre; OK is at right angles to PA', the longest line from P; DL, EM are projections of OK on BB', CC'; then

DL = EM, PD = PE = ½PK,

BP − B'P = 2PL = PK − 2DL,

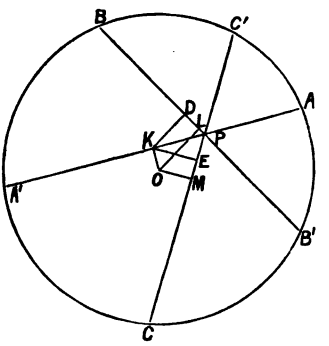
CP − C'P = 2PM = PK + 2EM,

AP − A'P = −2PK;

therefore

AP + BP + CP = A'P + B'P + C'P.

The maximum limiting case is when P is at O, and then the sum is $3r$. The minimum limiting case is when P is on the circumference, and one chord, as BB', becomes a tangent; then A'P and CP are each $r\sqrt{3}$ and BP, B'P, C'P, AP all vanish.



Digitized by Google

Note by CONSTANCE I. MARKS, B.A.

The two solutions furnished to this Question (*cf. Reprint*, New Series, Vol. VII., p. 91) are in perfect agreement, since the lower limit is actually reached only when P ceases to be *within* the circle and when one of the chords ceases to be a *chord* and becomes a *tangent*.

15628. (ROBERT W. D. CHRISTIE.)—If $x_1^2+1 = p_1 y^m$ for all integral values of m, then $x_n^2+1 = p_n y^m$ ad inf. Generalize.

E.g.: $239^2+1 = 2 \cdot 13^4$; then $85922^2+1 = 5 \cdot 17 \cdot 3041 \cdot 13^4$ ad inf.

Solution by Lt.-Col. ALLAN CUNNINGHAM, R.E.

The scope of this question is by no means clear; for, if $x_1^2+1 = p_1 y^m$ be true for one value of m, this involves

$$x_1^2+1 \equiv 0 \pmod{y, y^2, y^3, \ldots, y^{m-1}, y^m};$$

so that $x_1^2+1 = P_m y^m$ is true with the same x_1 (and different values of P_m) for all lesser positive integral values of m.

Further, the result $x_1^2+1 \equiv 0 \pmod{y}$ involves $x_m^2+1 \equiv 0 \pmod{y^m}$ for all positive integral values of m, the value of x_m being, however, usually different for each m. Along with each value of m there are always two (incongruous) values of x_m, each less than y^m.

The mode of obtaining these two (the least possible) values of x_m, *i.e.*, the roots of $x_m^2+1 \equiv 0 \pmod{y^m}$, from a given solution (x_1) of $x_1^2+1 \equiv 0 \pmod{y}$ is very fully explained and illustrated by examples in the present writer's paper on "Period Lengths of Circulates," in *Mess. of Math.*, Vol. XXIX., 1900, pp. 145 *et seq.*; a table of x_m for all primes $y \not> 200$ is also there given (p. 166).

When the two roots x_m have been found of $x_m^2+1 \equiv 0 \pmod{y^m}$, all the other roots, say $x_{m,n}$, are included in the formula $x_{m,n} = M \cdot y^m \pm x_m$, and the value of p_n (in the original Question) is then given, simply as the quotient $p_n = (x_{m,n}^2+1)/y^m$.

This seems to be the generalization asked for.

Ex.—Given $239^2+1 \equiv 0 \pmod{13^4}$ [see p. 166, *op. cit.*]; therefore

$$(3 \cdot 13^4 + 239)^2+1 \equiv 0, \quad i.e., \quad 85922^2+1 \equiv 0 \pmod{13^4}.$$

The PROPOSER solves the problem as follows:—

Let $x^2+1 = py^m$. Then

$$(x+y^m)^2+1 = (p+2x+y^m)\,y^m \quad \text{or} \quad \lambda^2+1 = Xy^m.$$

Similarly, if $x^2+r = py^m$, where r = any root of p, then

$$(x+y^m)^2+r = (p+2x+y^m)\,y^m, \quad i.e., \quad \lambda^2+r = Xy^m.$$

Ex.—Let $13^2+2 = 19 \cdot 3^2$; then

$$265^2+2 = 17^2 \cdot 3^5, \quad 2695^2+2 = 41 \cdot 3^{11}, \quad \ldots.$$

Similarly for any other prime.

Digitized by Google

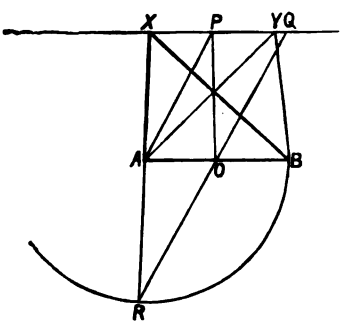
15509. (C. M. Ross.)—Having given the base and the altitude of a triangle, and that one of the angles at the base is double the other, show how to construct the triangle.

Solutions (I.) *by* J. H. Carson; (II.) *by* C. Searle.

(I.) At O, the mid-point of the given base AB, erect OP perpendicular to AB and make OP equal to the given altitude. Through P draw XPY parallel to AB. Join AP. With A for centre and AB for radius describe an arc of a circle BR. Through O draw QOR parallel to AP. Join RA, and produce it to meet XPY in X. Cut off

$$PY = PX.$$

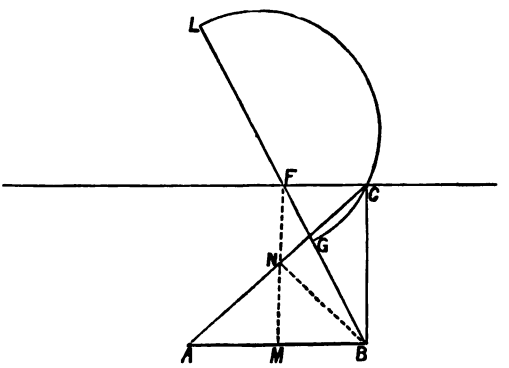
Then either ABX or ABY is the required triangle. For



$$XP : XA = PQ : AR = AO : AB = 1 : 2.$$

Thus AX = XY = YB, and the quadrilateral ABYX is clearly cyclic. Hence the angles XAY, YAB and ABX standing on equal arcs are equal; therefore $\angle XAB = 2\angle ABX$.

(II.) Let AB be the base, MF the altitude. Draw MF bisecting AB



at right angles. Divide FB at G and L, so that

$$FG : GB = FL : LB = 1 : 2.$$

On LG describe a semicircle cutting FC (parallel to AB) at C. Join CA, CB. Where CA cuts FM, viz. N, join BN. It can easily be shown that LCG is the locus of vertices of all triangles having the base FB and

Digitized by Google

sides CF, CB, such that CF : CB = 1 : 2; therefore CF : CB = AM : AB, CF : AM = CB : AB. But CF : AM = CN : NA; therefore

$$CB : AB = CN : NA;$$

therefore BN bisects $\angle ABC$. But the triangles AMN, BMN are equal in all respects; therefore

$$\angle MAN = \angle MBN = \tfrac{1}{2}\angle MBC \quad \text{or} \quad \angle ABC = 2\angle BAC.$$

12865. (W. Booth.)—If a plane be drawn touching the sphere $x^2+y^2+z^2 = R^2$ along its intersection with $z^2 = a^2x^2+b^2y^2$, prove that the equation of the developable so generated is obtained by equating to zero the discriminant of $x^2/(1-\lambda a^2)+y^2/(1-\lambda b^2)+z^2/(1+\lambda)-R^2 = 0$.

Solution by E. F. Tipple.

The surfaces $x^2(1-\lambda a^2)+y^2(1-\lambda b^2)+z^2(1+\lambda) = R^2$ intersect in the given curve. Reciprocating with respect to the sphere $x^2+y^2+z^2 = R^2$, we obtain that the surfaces $x^2/(1-\lambda a^2)+y^2/(1-\lambda b^2)+z^2/(1+\lambda) = R^2$ are enveloped by the developable into which the given curve reciprocates. But this developable is the envelope of the tangent planes to the sphere along the given curve. Hence developable is the envelope of the surfaces

$$x^2/(1-\lambda a^2)+y^2/(1-\lambda b^2)+z^2/(1+\lambda)-R^2 = 0.$$

15605. (James Blaikie, M.A.)—A. and B. play at lawn-tennis, and A.'s chance of winning each point is to B.'s chance as m to n. Find A.'s chance of winning the game (1) when the score stands at "deuce," (2) when it stands at A.'s vantage, (3) at B.'s vantage.

Solution by the Proposer *and* A. M. Nesbitt, M.A.

(1) If the score stand at deuce, the chance that A. will obtain vantage is $m/(m+n)$, that B. will obtain it $n/(m+n)$. In the next hand the chance that A. wins the game is clearly $m^2/(m+n)^2$, that B. wins it $n^2/(m+n)^2$; and thus the chance of deuce is $2mn/(m+n)^2$. Thus A.'s total chance of winning the game is

$$\frac{m^2}{(m+n)^2}\left[1+\frac{2mn}{(m+n)^2}+\frac{4m^2n^2}{(m+n)^4}+\dots\right] = \frac{m^2}{m^2+n^2}.$$

(2) If the score stand at A.'s vantage, his chance of winning in the next hand is $m/(m+n)$, while the chance of deuce is $n/(m+n)$; therefore A.'s chance of winning the game is

$$\frac{m}{m+n}+\frac{n}{m+n}\cdot\frac{m^2}{m^2+n^2} = \frac{m(m^2+mn+n^2)}{(m+n)(m^2+n^2)}.$$

Digitized by Google

(3) If the score stand at B.'s vantage, the chance of deuce in the next hand is $m/(m+n)$, and thus A.'s chance of winning the game is

$$\frac{m}{m+n} \cdot \frac{m^2}{m^2+n^2} = \frac{m^3}{(m+n)(m^2+n^2)}.$$

Example: When $m = n$, A.'s chance of winning is $\frac{1}{2}$ at deuce, $\frac{3}{4}$ at A.'s vantage, $\frac{1}{4}$ at B.'s vantage.

15645. (R. Chartres.)—The reciprocals of two consecutive numbers give the same number of recurring figures (n) in the scale radix (r), and the results looked upon as whole numbers are consecutive. Find the numbers and r and n.

Solution by James Blaikie, M.A., *and the* Proposer.

Let a, $a+1$ be the two consecutive numbers. Then, if $1/a = b/(r^n-1)$, we have $1/(a+1) = (b-1)/(r^n-1)$; therefore $r^n - 1 = ab = ab - a + b - 1$; therefore $b = a+1$, $r^n = a(a+1)+1$. Thus any value of a gives a corresponding value for r, if $n = 1$. For example, let $a = 2$, $r = 7$.

$$\tfrac{1}{2} = \cdot\dot{3} \text{ (scale of 7)}, \quad \tfrac{1}{3} = \cdot\dot{2} \text{ (scale of 7)};$$

therefore $r = 7$, $n = 1$ is one solution. Similarly, if $a = 3, 4, 5, 6, \ldots$, $r = 13, 21, 31, 43, \ldots$, $n = 1$. There is then given an infinite series of values of r corresponding to $n = 1$. It is clear that $a(a+1)+1$ cannot be a square, since it is intermediate in value between a^2 and $(a+1)^2$. Thus n cannot be even. If, however, we take $a = 18$, $r^n = 18 \times 19 + 1 = 343 = 7^3$. In the scale of 7, 18 is written 24. Thus we have

$$\tfrac{1}{24} = \cdot\dot{0}2\dot{5} \text{ (scale of 7)}, \text{ and } \tfrac{1}{25} = \cdot\dot{0}2\dot{4} \text{ (scale of 7)};$$

therefore $r = 7$, $n = 3$ is a solution. There seems no reason to doubt that the series $a(a+1)+1$, if continued, will give an infinite number of values of n, as well as an infinite number of values of r.

15608. (Lt.-Col. Allan Cunningham, R.E.)—Factorize into *prime* factors $N = (2^{64} \mp 2^{32} + 1)^2 + 1$. (Each number contains 39 figures.)

By the Proposer.

$$N = (2^{64}+1)^2 \mp 2 \cdot 2^{32}(2^{64}+1) + 2^{64} + 1 = 2(2^{64}+1)(2^{63} \mp 2^{32} + 1).$$

Hereof, $2^{64}+1 = 274177 \cdot 67280421310721$. [M. Landry's result.]

Now, let $L' = 2^{63} - 2^{32} + 1$, $M' = 2^{63} + 2^{32} + 1$, and let $N' = L'M'$. Then $N' = 2^{126} + 1$; so that L', M' are the *Aurifeuillian Factors* of N', which has been factorized by MM. Lelasseur and Aurifeuille: but, as the factors L', M' require now to be shown separately and complete, the reduction is here given.

Digitized by Google

Here N′ contains the algebraic factors $N_{18} = 2^{18}+1$, $N_{42} = 2^{42}\pm 1$, each of which is a *Bin-Aurifeuillian*. Resolving these into their Aurifeuillian Factors, say $N_{18} = L_{18}.M_{18}$, $N_{42} = L_{42}.M_{42}$, by the rules given in the writer's paper "On Aurifeuillians" (see *Proc. Lond. Math. Soc.*, Vol. XXIX, pp. 391 *et seq.*),

$L_{18} = 2^9 - 2^5 + 1 = 13.37$; $\quad M_{18} = 2^9 + 2^5 + 1 = 5.109$;
$L_{42} = 2^{21} - 2^{11} + 1 = 5.29.14449$; $\quad M_{42} = 2^{21} + 2^{11} + 1 = 13.113.14229$;

and, by the properties of Aurifeuillian Factors, the fact that M′ contains 5 shows that L′ must contain $\frac{1}{5}L_{18}.M_{42}$; M′ must contain $\frac{1}{5}M_{18}.L_{42}$. Hence, effecting the divisions,

$$\left.\begin{aligned} L' &= 13.37.118.1429.118750098349 \\ M' &= 5.109.29.14449.40388473189 \end{aligned}\right\};$$ this completes the resolution.

Similarly it may shown that

$$(2^8-2^4-1)^2+1 = 3.257.113; \quad (2^8+2^4+1)^2+1 = 2.257.(5.29);$$
$$(2^{16}-2^8+1)^2+1 = 2.65537.(13.41.61);$$
$$(2^{16}+2^4+1)^2+1 = 2.65537.(25.1321);$$
$$(2^{32}-2^{16}+1)^2+1 = 2.(641.6700417).(5581.384773);$$
$$(2^{32}+2^{16}+1)^2+1 = 2.(641.6700417)(5.8681.49477).$$

But the resolution cannot (at present) be carried beyond the numbers N given in the Question. For properties of Bin-Aurifeuillians required above, see the writer's Paper above quoted, pp. 391, 406, 407.

15142. (Professor NANSON.)—If $x+y+z+\omega = 0$, $ax+by+cz+d\omega = 0$,
$(a-d)^2(b-c)^2(x\omega+yz)+(b-d)^2(c-a)^2(y\omega+zx)$
$+(c-d)^2(a-b)^2(z\omega+xy) = 0$,
prove that
$x : y : z : \omega = (d-b)(d-c)(b-c) : (d-c)(d-a)(c-a)$
$: (d-a)(d-b)(a-b) : (b-c)(c-a)(a-b)$.

Solution by the PROPOSER.

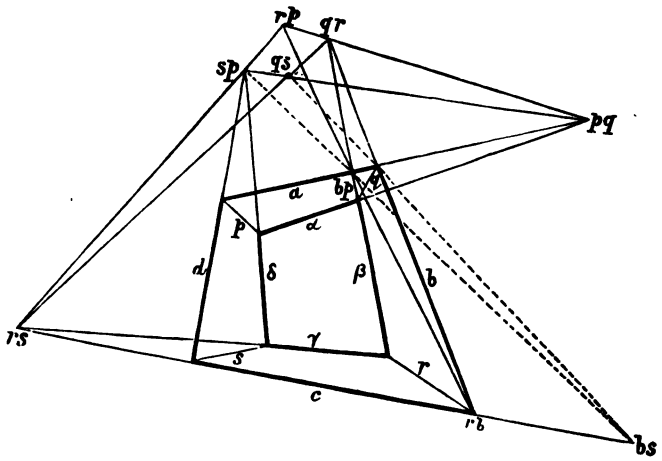
Let $Ax = \xi + a\eta + a^2\zeta + a^3\omega$, ..., where $A = (a-b)(a-c)(a-d)$, ...; then from the two linear equations we get $\omega = 0$, $\zeta = 0$, and hence the quadratic takes the form $P\xi^2 + Q\xi\eta + R\eta^2 = 0$, in which $P = 0$, $Q = 0$; so that we get the two-fold solution $Ax = By = Cz = D\omega$.

15602. (L. ISSERLIS.)—The corners ab, bc, cd, da of a quadrilateral whose sides are a, b, c, d are joined by lines q, r, s, p to the corners $a\beta$, $\beta\gamma$, $\gamma\delta$, $\delta\alpha$ of a quadrilateral whose sides are α, β, γ, δ. a, α meet in pq, b, β in qr, c, γ in rs, and d, δ in sp. The lines rs, sp and qr, pq meet in pr; rp, rb, and a meet in bp; and rs, qr and sp, pq in qs. Prove that the lines qs, $b\alpha$ and ps, bp meet on c.

Digitized by Google

Solution by the PROPOSER.

By a well known theorem, if a, b, c are three bodies moving in one plane and if bc, ca, ab are the instantaneous centres of the relative motion of b and c, c and a, and a and b, then they are in a straight line. Now consider a jointed system consisting of twelve bars $abcd$, $\alpha\beta\gamma\delta$, $pqrs$. $2n-3$ bars joining n points will form a rigid framework. For eight points $2n-3$ is 13; but we have only twelve bars. Hence, when one bar is fixed and the motion of any joint is given, the motion of all the others is uniquely determined. The rider becomes self-evident if the notation xy, say, is understood to mean the instantaneous centre of the rod x with regard to the rod y. For the construction simply determines the instantaneous centre bs as the intersection of the dotted lines; but bs must lie on the same straight line as rs and rb, which is c.



To find rs, produce c and γ to meet. To find bs, (1) a, α meet in pq, b, β in qr, d, δ in sp; (2) rs, sp meets qr, pq in rp; (3) rp, rb meets a (which is ap, ab) in bp; (4) rs, qr meets sp, pq in qs. Therefore bs lies (A) on qs, bq; (B) on ps, bp, and therefore these two meet in a point; (C) on c.

4950. (Professor CLIFFORD, F.R.S.)—Prove that every matrix of the second order may be expressed in the form $aI + bJ$, where I is the matrix unity and J a matrix such that $J^2 = -I$. Hence find an expression for any power of such a matrix. (See Cayley on "Matrices," *Phil. Trans.*, 1858.) Required a geometrical representation for a non-self-conjugate linear and vector function.

Digitized by Google

Solution by Professor S. SIRCOM.

Let M denote the matrix $\begin{pmatrix} \alpha, & \beta \\ \gamma, & \delta \end{pmatrix}$; then we may write

$$M = \begin{pmatrix} \frac{1}{2}(\alpha+\delta), & 0 \\ 0, & \frac{1}{2}(\alpha+\delta) \end{pmatrix} + \begin{pmatrix} \frac{1}{2}(\alpha-\delta), & \beta \\ \gamma, & \frac{1}{2}(\delta-\alpha) \end{pmatrix} = \tfrac{1}{2}(\alpha+\delta)\,I + L,$$

where $$L^2 = [\tfrac{1}{4}(\alpha-\delta)^2 + \beta\gamma]\begin{pmatrix} 1, & 0 \\ 0, & 1 \end{pmatrix}$$

by the law of formation;

$$\begin{pmatrix} \alpha, & \beta \\ \gamma, & \delta \end{pmatrix}^2 = \begin{pmatrix} \alpha^2+\beta\gamma, & \beta(\alpha+\delta) \\ \gamma(\alpha+\delta), & \beta\gamma+\delta^2 \end{pmatrix}.$$

Hence, if $J = \begin{pmatrix} (\alpha-\delta)/R, & 2\beta/R \\ 2\gamma/R, & (\delta-\alpha)/R \end{pmatrix}$, where $R = \sqrt{[-4bc-(\alpha-\delta)^2]}$, then $J^2 = -I$; therefore $M = \frac{1}{2}(\alpha+\delta)\,I + \frac{1}{2}RJ$, and a and b are determined.

Writing $M = r\cos qI + r\sin qJ$, we have

$$M^2 = r^2(\cos^2 qI^2 + \cos q\sin qIJ + \sin q\cos qJI + \sin^2 qJ^2).$$

But $I^2 = I$, $IJ = JI = J$, $J^2 = -I$, whence $M^2 = r^2\cos 2qI + r^2\sin 2qJ$, and generally $M^n = r^n\cos nqI + r^n\sin nqJ$, where $r^2 = a^2+b^2 = a\delta-\beta\gamma$, the determinant of the matrix, and $r\cos q = \frac{1}{2}(\alpha+\delta)$, q being the angle given in § 47 of Cayley's paper.

Hence, if z is the point $x+iy$, Mz is a linear and vector function of z, and is constructed by drawing from the point $\frac{1}{2}(\alpha+\delta)z$ a vector equal to $\frac{1}{2}R$ times the vector Jz; then the point z, so determined, is the linear and vector function of z required.

15644. (W. H. BLYTHE, M.A.)—If M divides the period of 1/M, or, in other words, if 1/M and $1/M^2$ have the same period, then $M = 2(3^n)+1$, where n is a positive integer, and a multiple of 5, including zero. Is this the case? It is to be observed that, being true for 3 and 487, it is true also for their product 1461.

Note by Lt.-Col. ALLAN CUNNINGHAM, R.E.

The suggested theorem that, "If 1/M and $1/M^2$ have the same period, then $M = 2(3^n)+1$" is certainly not (necessarily) true when M is composite. In fact, the example given disproves it. For

$$10^{486} \equiv +1 \pmod{3\cdot 487 \text{ and } 3^2\cdot 487^2},$$

and yet $3\cdot 487 \neq 2\cdot 3^n+1$.

It is believed that there is no known direct method of finding numbers M such that $r \equiv 1 \pmod{p \text{ and } p^2}$, when $r < p$ and $l \not> p-1$.

The writer gave a list of 26 such cases in his paper on "Period Lengths of Circulates" in *Mess. of Math.*, Vol. XXIX., 1900, p. 158. To these may now be added

$$252^{12} \equiv 1 \pmod{997 \text{ and } 997^2}, \quad 390112^4 \equiv \pmod{17^5 \text{ and } 17^6}.$$

Digitized by Google

When two such cases can be found for the same base (r) with different prime moduli (p_1, p_2), then these give a case with the composite modulus $M = p_1 p_2$. Thus, from the writer's 26 cases above quoted, it follows that

$18^{36} \equiv 1 \pmod{7^2 . 37 \text{ and } 7^3 . 37^2}$, $19^{42} \equiv 1 \pmod{7^2 . 43 \text{ and } 7^3 . 43^2}$.

The difficulty in finding such cases consists in the conditions $r < p$ and $l \not> p-1$. Without these limitations it would be easy to find cases when $r > p$ and $l > p-1$.

[In the writer's table above quoted, p. 158, 27 cases are printed; but one of these, viz., $44^{13} = 1 \pmod{53 \text{ and } 53^2}$, is erroneous.]

15583. (Rev. T. ROACH, M.A.)—Simplify

$$[(1-e^{i\theta}\cos\theta)/(1-e^{-i\theta}\cos\theta)]^{\frac{1}{2}i}.$$

Solutions (I.) *by* JAMES BLAIKIE, M.A., F. W. REEVES, B.A., *and others*; (II.) *by* C. M. ROSS.

(I.)

$$\left(\frac{1-e^{i\theta}\cos\theta}{1-e^{-i\theta}\cos\theta}\right)^{\frac{1}{2}i} = \left(\frac{1-(\cos\theta+i\sin\theta)\cos\theta}{1-(\cos\theta-i\sin\theta)\cos\theta}\right)^{\frac{1}{2}i} = \left(\frac{\sin^2\theta-i\sin\theta\cos\theta}{\sin^2\theta+i\sin\theta\cos\theta}\right)^{\frac{1}{2}i}$$

$$= \left(\frac{\sin\theta-i\cos\theta}{\sin\theta+i\cos\theta}\right)^{\frac{1}{2}i} = \left(\frac{e^{-i(\frac{1}{2}\pi-\theta)}}{e^{i(\frac{1}{2}\pi-\theta)}}\right)^{\frac{1}{2}i}$$

$$= (e^{-2i(\frac{1}{2}\pi-\theta)})^{\frac{1}{2}i} = e^{(\frac{1}{2}\pi-\theta)}.$$

(Other values may be found owing to the periodic character of $\sin\theta - i\cos\theta$.)

(II.)

$$\frac{1-e^{i\theta}\cos\theta}{1-e^{-i\theta}\cos\theta} = \frac{1-(\cos\theta+i\sin\theta)\cos\theta}{1-(\cos\theta-i\sin\theta)\cos\theta} = \frac{\sin\theta-i\cos\theta}{\sin\theta+i\cos\theta}$$

$$= \sin^2\theta-\cos^2\theta-2i\sin\theta\cos\theta = -(\cos 2\theta+i\sin 2\theta)$$

$$= \cos(\pi-2\theta)-i\sin(\pi-2\theta) = e^{-(\pi-2\theta)i}.$$

Hence $\left(\frac{1-e^{i\theta}\cos\theta}{1-e^{-i\theta}\cos\theta}\right)^{\frac{1}{2}i} = e^{-(\pi-2\theta)\frac{1}{2}i^2} = e^{\frac{1}{2}\pi-\theta} = \cosh(\frac{1}{2}\pi-\theta)+\sinh(\frac{1}{2}\pi-\theta)$.

15630. (Professor NEUBERG.)—Résoudre les systèmes

(1) $\sin x = a\cos y$, $\sin y = b\cos z$, $\sin z = c\cos x$;

(2) $\sin x = a\tan y$, $\sin y = b\tan z$, $\sin z = c\tan x$.

Solution by C. M. ROSS *and* R. TUCKER, M.A.

(1) $\sin^2 x = a^2(1-\sin^2 y) = a^2[1-b^2(1-\sin^2 z)]$

$= a^2[1-b^2(1-c^2\cos^2 x)] = a^2-a^2b^2+a^2b^2c^2(1-\sin^2 x)$;

therefore $(1+a^2b^2c^2)\sin^2 x = a^2(1-b^2+b^2c^2)$;

therefore $x = n\pi \pm \sin^{-1} a[(1-b^2+b^2c^2)/(1+a^2b^2c^2)]^{\frac{1}{2}}$.

Digitized by Google

Again, $a^2\cos^2 y = a^2(1-b^2+b^2c^2)/(1+a^2b^2c^2)$;

therefore $\sin^2 y = b^2(1-c^2+c^2a^2)/(1+a^2b^2c^2)$;

therefore $y = n\pi \pm \sin^{-1} b\,[(1-c^2+c^2a^2)/(1+a^2b^2c^2)]^{\frac{1}{2}}$.

Similarly, $z = n\pi \pm \sin^{-1} c\,[(1-a^2+a^2b^2)/(1+a^2b^2c^2)]^{\frac{1}{2}}$.

(2) $$\sin^2 x = \frac{a^2\sin^2 y}{1-\sin^2 y} = \frac{a^2b^2\tan^2 z}{1-b^2\tan^2 z} = \frac{a^2b^2\sin^2 z}{1-(1+b^2)\sin^2 z}$$
$$= \frac{a^2b^2c^2\tan^2 x}{1-(1+b^2)c^2\tan^2 x} = \frac{a^2b^2c^2\sin^2 x}{1-[1+c^2(1+b^2)]\sin^2 x};$$

therefore $\sin^2 x = (1-a^2b^2c^2)/[1+c^2(1+b^2)]$, $\sin^2 x = 0$;

therefore $x = n\pi \pm \sin^{-1}\{(1-a^2b^2c^2)/[1+c^2(1+b^2)]\}^{\frac{1}{2}}$ and $n\pi$.

Again, $a^2\tan^2 y = (1-a^2b^2c^2)/[1+c^2(1+b^2)]$;

therefore $a^2[1+c^2(1+b^2)] = (1-a^2b^2c^2)(\operatorname{cosec}^2 y - 1)$;

therefore $\sin^2 y = (1-a^2b^2c^2)/[1+a^2(1+c^2)]$;

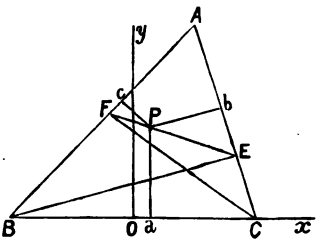
therefore $y = n\pi \pm \sin^{-1}\{(1-a^2b^2c^2)/[1+a^2(1+c^2)]\}^{\frac{1}{2}}$ and $n\pi$.

Similarly $z = n\pi \pm \sin^{-1}\{(1-a^2b^2c^2)/[1+b^2(1+a^2)]\}^{\frac{1}{2}}$ and $n\pi$.

15565. (Hon. G. R. Dick.)—If the bisectors of the base angles B and C of a triangle ABC meet the opposite sides in E and F, BC being given and the angle A constant, the line EF will touch a conic whose eccentricity is $\frac{1}{2}\operatorname{cosec}\frac{1}{2}A$.

Solution by the Proposer.

It is easily proved by trilinear co-ordinates, or otherwise, that, if P be any point taken within the triangle ABC and on the line EF, and Pa, Pb, Pc are the perpendiculars let fall on the sides, then Pa = Pb + Pc. Take now for axes of Cartesian co-ordinates OC and a perpendicular to OC at O; then



Pa = y,

Pb = $\frac{1}{2}a\sin C - x\sin C - y\cos C$,

Pc = $\frac{1}{2}a\sin B + x\sin B - y\cos B$;

therefore

$$y = \tfrac{1}{2}a(\sin B + \sin C) + x(\sin B - \sin C) - y(\cos B + \cos C).$$

This relation holds good for all points between E and F, and therefore for all points whatever on EF—is, in fact, the equation of EF. Now put $B - C = 2\theta$ and we get

$$y = a\cos\tfrac{1}{2}A\cos\theta + 2x\sin\theta\tfrac{1}{2}A - 2y\sin\tfrac{1}{2}A\cos\theta$$
$$= \cos\theta(a\cos\tfrac{1}{2}A - 2\sin\tfrac{1}{2}A\,y) + 2x\sin\theta\sin\tfrac{1}{2}A.$$

Thus the envelope of EF is $y^2 = (a\cos\frac{1}{2}A - 2y\sin\frac{1}{2}A)^2 + 4x^2\sin^2\frac{1}{2}A$, or

$$4x^2\sin^2\tfrac{1}{2}A + y^2(4\sin^2\tfrac{1}{2}A - 1) - 2a\sin A\,y + a^2\cos^2\tfrac{1}{2}A = 0,$$

Digitized by Google

the equation of a conic whose eccentricity is $\frac{1}{2}\operatorname{cosec}\frac{1}{2}A$. The focal axis is perpendicular to BC; the co-ordinates of the centre are

$$x = 0, \quad y = a \sin A/(1-2\cos A).$$

15526. (Professor Neuberg.)—Soient M un point fixe et N un point mobile d'une hyperbole équilatère. Trouver le lieu du point d'intersection de la normale en N à l'hyperbole avec la perpendiculaire élevée en M sur la corde MN.

Solutions by S. C. Bagchi, B.A., *and* C. M. Ross.

L'équation d'une hyperbole équilatère rapportée à ses asymptotes est $xy = \kappa^2$. Soient M et N (x_0, y_0), (x_1, y_1) respectivement. Si (ξ, η) est le point d'intersection de la normale en N à l'hyperbole avec la perpendiculaire élevée en M sur la corde MN, l'on aura le lieu du point (ξ, η) en éliminant x_1 et y_1 entre

$$(\xi - x_1)x_1 = (\eta - y_1)y_1 \quad \text{.....(i.)},$$
$$(y_0 - y_1)(\eta - y_0) = (x_1 - x_0)(\xi - x_0) \quad \text{.....(ii.)},$$
$$x_1 y_1 = x_0 y_0 = \kappa^2 \quad \text{.....(iii.)}.$$

Les relations (ii.) et (iii.) donnent

$$y_1(\eta - y_0) = x_0(\xi - x_0), \quad x_1(\xi - x_0) = y_0(\eta - y_0),$$

et on a pour le lieu, tenant compte de la relation (i.),

$$(\xi - x_0)[\xi y_0(\eta - y_0)^3 + x_0^2(\xi - x_0)^3] = (\eta - y_0)[\eta x_0(\xi - x_0)^3 + y_0^2(\eta - y_0)^3].$$

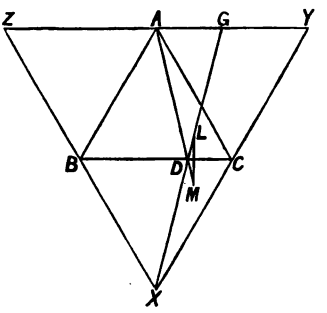
10349. (Professor Hain.)—Étant donné un triangle équilatéral ABC, si M, N, P sont les symétriques d'un point quelconque L par rapport aux trois côtés, démontrer que les droites AM, BN, CP se rencontrent en un même point.

Additional Solution by W. F. Beard, M.A.

Let AM, BN, CP meet the sides of ABC at D, E, F. Let X, Y, Z be the images A, B, C with regard to BC, CA, AB; then it is plain that AM, XL meet at D. Produce XL, YL, ZL to meet YZ, ZX, XY at G, H, K. Then, since YZ is parallel to BC,

$$BD : DC = ZG : GY,$$

and similar results are obtained in the other cases.



But, by Ceva's theorem,

$$ZG.YK.XH = GY.KX.HZ;$$

therefore

$$BD.CE.AF = DC.EA.FB;$$

and therefore AD, BE, CF are concurrent,

Digitized by Google

INDEX.

The numbers refer to the pages, and in the case of Questions and Solutions the reference is, in general, to the page bearing the Question.

(*a*)—SUBJECTS.

Algebra (General) 21, 28, 30, 35, 40, 41, 43, 47, 49, 50, 55, 69, 78, 88, 93, 94, 95, 106, 107.
Algebra (Higher), 39, 40, 50, 55, 78, 92, 95, 97, 108.
Arithmetic and Theory of Numbers, 29, 38, 43, 52, 56, 60, 62, 68, 72, 79, 81, 103, 106, 109.
Attractions, 46.
Calculus, Differential and Integral, 26, 48, 87, 98.
Combinations and Permutations, 77.
Co-ordinate Geometry of Surfaces whose Algebraic Equations are of the first or second degree, 34, 42, 59, 61, 92, 105.
Co-ordinate Geometry, Plane—of Point, Straight Line, Circle, and Conic—(Problems connected with), 17, 24, 26, 31, 36, 37, 51, 52, 56, 61, 63, 63, 67, 68, 72, 76, 76, 86, 93, 95, 97, 100, 111, 112.
Curves, Plane (exclusive of Circle and Conic), 17, 24, 28, 30, 31, 37, 49, 53, 57, 63, 70, 75, 77, 100, 112.
Curves, Space, 105.
Curves, Straight Lines, and Points (given specifically as the intersections of surfaces), 34, 105.
Geometrical Conics, 32, 48, 54, 69, 73, 85, 86, 87, 100.
Geometry, Elementary—of Straight Lines in a Plane, of Plane Rectilineal Figures, and of the Circle, 20, 25, 33, 40, 55, 64, 65, 74, 104.
Geometry, Higher and Modern, 17, 24, 28, 30, 31, 33, 37, 48, 49, 55, 58, 59, 71, 90, 91, 95, 101, 102, 107, 108, 112.
Geometry in *n* dimensions, 34.
Hyperbolic Functions, Direct and Inverse, 74, 98, 110.
Integrals, Definite (Problems in or solved by), 23, 46, 48.
Maxima and Minima, 33, 65, 66, 70, 75, 91, 97, 102.
Mean Values, 32.
Mechanics of Particles and Rigid Bodies, 56, 94, 99, 108.
Minima and Maxima, 33, 65, 66, 70, 75, 91, 97, 102.
Permutations and Combinations, 77.
Points, Straight Lines, and Curves (given specifically as the intersections of surfaces), 34, 105.
Probabilities, 42, 105.
Series, 22, 23, 29, 35, 41, 48, 58, 63, 74, 87, 91.
Solids bounded by Plane Faces (Problems connected with), 67.
Straight Lines, Curves, and Points (given specifically as the intersections of surfaces), 34.
Surfaces (exclusive of those whose Algebraic Equations are of the first or second degree), 61, 65, 75, 92, 105.
Theory of Numbers and Arithmetic, 29, 38, 43, 52, 56, 60, 62, 68, 72, 79, 81, 103, 106, 109.
Transformations, Geometrical (of special types), 18, 71.
Trigonometry (Problems in or solved by), 20, 23, 24, 26, 41, 57, 59, 61, 65, 66, 74, 91, 95, 98, 110, 110, 111.

Digitized by Google

(*b*)—CONTRIBUTORS.

Anderson, Rev. W., M.A., 20.
Archibald, R. C., M.A., Ph.D., 53.
Arnold, I., 70.
Bagchi, S. C., B.A., 20, 112.
Barniville, J. J., B.A., 35, 91.
Bateman, H., B.A., 17.
Bates, G. N., M.A., 24, 37.
Beard, W. F., M.A., 54, 73, 86, 112.
Bell, A. H., 88.
Bickerdike, C., 88, 100.
Biddle, D., 29, 33, 65, 70, 74, 88.
Blaikie, J., M.A., 21, 25, 48, 50, 59, 60, 68, 70, 74, 77, 78, 88, 105, 106, 110.
Blythe, W. H., M.A., 88, 109.
Booth, W., 105.
Brown, Prof. J., 74.
Bruce, W. R., 98.
Carson, J. H., 104.
Catalá, Prof. L., 40.
Cave, F. E., 42, 46, 52, 57, 61, 68, 72, 99.
Césaro, E., 57.
Chartres, R., 29, 32, 35, 42, 50, 55, 56, 87, 106.
Christie, R. W. D., 52, 53, 79, 88, 103.
Clifford, Prof., F.R.S., 30, 59, 108.
Collins, M , B.A., 28.
Crofton, Prof. M. W., F.R.S., 32.
Cullen, Rev. J., 24, 92.
Cunningham, Lt.-Col. A., R.E., 21, 28, 30, 38, 43, 49, 50, 60, 62, 68, 72, 78, 79, 88, 97, 103, 106, 109.
Davis, R. F., M.A., 23, 37, 47, 48, 54, 71, 74, 86, 98.
Dick, Hon. G. R., 61, 65, 111.
Durell, C. V., 26.
Escott, Prof. E. B., 81.
Garrett, C. A. B., 68.
Genese, Prof. R. W., M.A., 46.
Gordon, A., 42.
Graves, H. G., 67.
Greenstreet, W. J., M.A., 30, 49, 77, 94.
Hain, Prof., 59, 112.
Hatton, J. L. S., 55.
Heppel, G., M.A., 91, 102.
Holden, A., B.A., 90.
Hudson, Prof. W. H. H., M.A., 22, 77.
Isserlis, L., B.A., 72, 97, 107, 108.
Kent, H. C., 75.
Knowles, R., B.A., 72.
Lauvernay, Prof., 55.
Lévy, Prof., 94.
Marks, C. I., B.A., 91, 103.
Morley, Prof., 34.
Muir, T., LL.D., F.R.S., 50, 97.
Nanson, Prof., 30, 31, 34, 39, 48, 50, 57, 58, 60, 63, 67, 70, 76, 97, 107.
Nash, Prof., M.A., 17.
Nesbitt, A. M., M.A., 17, 26, 27, 41, 50, 56, 63, 67, 68, 85, 86, 93, 93, 94, 95, 100, 101, 105.
Neuberg, Prof., 49, 61, 100, 101, 110, 112.
Orchard, Prof. H. L., M.A., B.Sc., 58, 94.
Plamenewski, Prof., M.A., 26.
Porter, F., M.A., 21, 64.
Prescott, J., B.A., 65.
Rau, Prof. Hanumanta, 41.
Rau, K. Deva, B.A., 24, 54, 58, 86, 100.
Reay, L. E., B.A., 86.
Reeves, F. W., B.A., 68, 110.
Richardson, A. R., 32, 52.
Roach, Rev. T., M.A., 20, 22, 66, 98, 110.
Ross, C. M., 23, 30, 38, 40, 47, 55, 58, 64, 75, 78, 90, 91, 104, 110, 110, 112.
Russo, Prof., 76.
Sanjána, Prof., M.A., 17, 21, 23, 31, 33, 48, 52, 54, 55, 61, 63, 6., 68, 69, 74, 78, 86, 93, 95, 96, 101.
Sarkar, Prof. Nilkantha, M.A., 99.
Scott, G., M.A., 70.
Scrimgeour, W., M.A., B.Sc., 67, 98.
Searle, C., 104.
Simmons, Rev. T. C., M.A., 42.
Sircom, Prof. S., 36, 53, 90, 109.
Taylor, H. M., M.A., F.R.S., 31.
Taylor, J. H., M.A., 25.
Terry, Rev. T. R., M.A., 48.
Tipple, E. F., 105.
Tucker, R., M.A., 20, 25, 38, 68, 69, 74, 86, 95, 100, 110.
Wallis, B. C., F.C.P., 70.
Whitehead, R. F., B.A., 24.
Whitworth, Rev. Preb. W. A., M.A., 43.
Wiggins, Rev. T., B.A., 24. 43, 56, 58, 60, 66, 75.
Woodall, H. J., A.R.C.S., 28, 63.
Youngman, C. E., M.A., 65, 73.
Zerr, Prof. G. B. M., 41, 75, 76.

A. W. T., 97.
θ. β., 28, 36.
"Solidus," 29, 47, 70.
"Stranger," 86.

Printed by C. F. Hodgson & Son, 2 Newton Street, High Holborn, W.C.

Digitized by Google

Digitized by Google

Digitized by Google